Quiz #4 (EngMath I) [Monday, Nov. 21, 2016]

Name: ______________ **Dept:** ______________ **ID No:** ______________

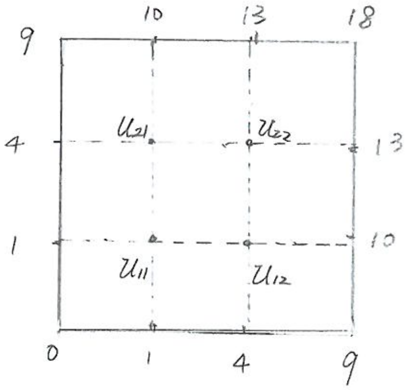
1. (15 points) Using $h = k = 1$, approximate the solution to the following elliptic partial differential equation

$$u_{xx} + 4u_{yy} = xy, \quad 0 \le x \le 3,\ 0 \le y \le 3$$

with boundary conditions:

$$u(x,0) = x^2, \quad u(x,3) = x^2 + 9, \quad 0 \le x \le 3;$$
$$u(0,y) = y^2, \quad u(3,y) = 9 + y^2, \quad 0 \le y \le 3.$$

Set up a system of linear equations.



$$\frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2} + 4\times\frac{U_{i,j+1} - 2U_{ij} + U_{i,j-1}}{k^2} = ij$$

$\because h^2k^2 = 1$

$\therefore U_{i+1,j} - 2U_{ij} + U_{i-1,j} + 4U_{i,j+1} - 8U_{ij} + 4U_{i,j-1} = ij$

$U_{i+1,j} - 10U_{ij} + U_{i-1,j} + 4U_{i,j+1} + 4U_{i,j-1} = ij$

$$\begin{cases} U_{21} - 10U_{11} + U_{01} + 4U_{12} + 4U_{10} = 1 \\ U_{22} - 10U_{12} + U_{02} + 4U_{13} + 4U_{11} = 2 \\ U_{31} - 10U_{21} + U_{11} + 4U_{22} + 4U_{20} = 2 \\ U_{32} - 10U_{22} + U_{12} + 4U_{23} + 4U_{21} = 4 \end{cases} \Rightarrow \begin{cases} -10U_{11} + 4U_{12} + U_{21} = -4 \\ 4U_{11} - 10U_{12} + U_{22} = -42 \\ U_{11} - 10U_{21} + 4U_{22} = -24 \\ U_{12} + 4U_{21} - 10U_{22} = -61 \end{cases}$$

$$\begin{bmatrix} -10 & 4 & 1 & 0 \\ 4 & -10 & 0 & 1 \\ 1 & 0 & -10 & 4 \\ 0 & 1 & 4 & -10 \end{bmatrix} \begin{bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{bmatrix} = \begin{bmatrix} -4 \\ -42 \\ -24 \\ -61 \end{bmatrix}$$

2. (5 points) Compute the following integral numerically using the Gauss quadrature with $n = 3$:

$$\int_3^7 \frac{dx}{\sqrt{x^2+3}}$$

$x = at + b$

$\begin{cases} 7 = a+b \\ 3 = -a+b \end{cases} \Rightarrow \begin{array}{l} x = 2t+5 \\ dx = 2dt \end{array}$ (+2)

$$\int_3^7 \frac{dx}{\sqrt{x^2+3}} = \int_{-1}^{1} \frac{2dt}{\sqrt{(2t+5)^2+3}} = 2\times\left(\frac{5}{9}\times\frac{1}{\sqrt{(5-2\times\sqrt{\frac{3}{5}})^2+3}} + \frac{8}{9}\times\frac{1}{\sqrt{28}} + \frac{5}{9}\,\frac{1}{\sqrt{(5+2\sqrt{\frac{3}{5}})^2+3}}\right)$$ (+3)

3. (10 points) Find the cubic spline $g(x)$ to the following data, with $k_0 = -1$ and $k_3 = 1$:

$$f_0 = f(0) = 0, f_1 = f(1) = 0, f_2 = f(2) = -1, f_3 = f(3) = 1.$$

$\begin{cases} k_0+4k_1+k_2 = \frac{3}{1}(f_2-f_0) = -3 \\ k_1+4k_2+k_3 = \frac{3}{1}(f_3-f_1) = 3 \end{cases} \Rightarrow \begin{cases} k_1 = -\frac{2}{3} \\ k_2 = \frac{2}{3} \end{cases}$ (+3)

$P_0(x) = ax^3+bx^2+cx+d$

$\begin{cases} P_0(0)=0 \\ P_0(1)=0 \\ P_0'(0)=-1 \\ P_0'(1)=-\frac{2}{3} \end{cases} \Rightarrow \begin{cases} a=-\frac{5}{3} \\ b=\frac{8}{3} \\ c=-1 \\ d=0 \end{cases} \Rightarrow P_0(x) = -\frac{5}{3}x^3+\frac{8}{3}x^2-x \quad (0\le x\le 1)$ (+2)

$P_1(x) = a(x-1)^3+b(x-1)^2+c(x-1)+d$

$\begin{cases} P_1(1)=0 \\ P_1(2)=-1 \\ P_1'(1)=-\frac{2}{3} \\ P_1'(2)=\frac{2}{3} \end{cases} \Rightarrow \begin{cases} a=2 \\ b=-\frac{7}{3} \\ c=-\frac{2}{3} \\ d=0 \end{cases} \Rightarrow P_1(x) = 2(x-1)^3-\frac{7}{3}(x-1)^2-\frac{2}{3}(x-1) \quad (1\le x\le 2)$ (+2)

$P_2(x) = a(x-2)^3+b(x-2)^2+c(x-2)+d$

$\begin{cases} P_2(2)=-1 \\ P_2(3)=1 \\ P_2'(2)=\frac{2}{3} \\ P_2'(3)=1 \end{cases} \Rightarrow \begin{cases} a=-\frac{7}{3} \\ b=\frac{11}{3} \\ c=\frac{2}{3} \\ d=-1 \end{cases} \Rightarrow P_2(x) = -\frac{7}{3}(x-2)^3+\frac{11}{3}(x-2)^2+\frac{2}{3}(x-2)-1 \quad (2\le x\le 3)$

(+3)