Engineering Mathematics I
(Comp 400.001)

Final Exam: June 19, 2001

1. (15 points) Show that the given integral represents the indicated function.

$$\int_0^\infty \left[\left(\frac{\sin 2\omega}{\omega}\right)\cos \omega x + \left(\frac{1-\cos 2\omega}{\omega}\right)\sin \omega x\right] d\omega = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi & \text{if } 0 < x < 2 \\ \pi/2 & \text{if } x = 2 \\ 0 & \text{if } x > 2 \end{cases}$$

Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \pi & \text{if } 0 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$$

$$\Rightarrow \quad A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv = \frac{1}{\pi}\int_0^2 \pi\cdot\cos wv\,dv$$

$$= \left[+\frac{1}{w}\sin wv\right]_0^2 = \frac{\sin 2w}{w}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv = \int_0^2 \sin wv\,dv$$

$$= \left[-\frac{1}{w}\cos wv\right]_0^2 = \frac{1-\cos 2w}{w}$$

2. (15 points)

(a) Find the Fourier series of the following periodic function:

$$f(x+2\pi) = f(x) = x^2, \quad \text{for } -\pi < x < \pi$$

(b) Using this result, show that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{12}$$

(a) $f(x)$: even function

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}x^2 \sin nx \Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x \sin nx \, dx \right]$$

$$= -\frac{4}{n\pi}\left[-\frac{1}{n}x \cos nx \Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx \right]$$

$$= \frac{4\pi}{n^2\pi}\cos n\pi - \frac{4}{n^2\pi}\left[\frac{1}{n}\sin nx\right]_0^{\pi}$$

$$= \frac{4}{n^2}(-1)^n$$

$$f(x) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cdot \cos nx$$

(b) $$0 = f(0) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$\sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^{n+1} = \frac{1}{3}\pi^2$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{12}\pi^2$$

Quiz #7 (CSE 400.001)

Thursday, June 7, 2001

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Dept: ____________________ **ID No:** ____________________

1. (7 points) Compute the Fourier series of the following periodic function, of period $p = 2L = 2$:

$$f(x) = x + x^2, \quad -1 < x < 1.$$

$f(x) = f_1(x) + f_2(x)$, where $f_1(x) = x$, $f_2(x) = x^2$

$f_1(x) = x$: odd $\Rightarrow$ $b_n = 2\int_0^1 x \sin n\pi x \, dx = -\frac{2}{n\pi}\cos n\pi$

$\therefore f_1(x) = \frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \cdots\right)$

+2

+3

$f_2(x) = x^2$: even $\Rightarrow$ $a_0 = \int_0^1 x^2 dx = \frac{1}{3}$

$a_n = 2\int_0^1 x^2 \cos n\pi x \, dx = \frac{4}{(n\pi)^2}\cos n\pi$

$\therefore f_2(x) = \frac{1}{3} + \frac{4}{\pi^2}\left(-\cos \pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \cdots\right)$

+2

$$\begin{aligned} f(x) &= f_1(x) + f_2(x) \\ &= \frac{1}{3} + \frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \cdots\right) \\ &\quad + \frac{4}{\pi^2}\left(-\cos \pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \cdots\right) \end{aligned}$$

2. (8 points) Compute the Fourier transform of the following function:

$$f(x) = \begin{cases} x + x^2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (x+x^2) e^{-iwx} dx \quad +1$$

$$\int_{-1}^{1} (x+x^2) e^{-iwx} dx = \frac{1}{-iw} (x+x^2) e^{-iwx} \Big|_{-1}^{1} + \frac{1}{iw} \int_{-1}^{1} (1+2x) e^{-iwx} dx \quad +2$$

$$= \frac{1}{-iw} 2e^{-iw} + \frac{1}{iw} \left\{ \frac{1}{-iw} (1+2x) e^{-iwx} \Big|_{-1}^{1} + \frac{1}{iw} \int_{-1}^{1} 2e^{-iwx} dx \right\} \quad +2$$

$$= \frac{2i}{w} e^{-iw} + \frac{1}{w^2} \left(3e^{-iw} + e^{iw} \right) + \frac{1}{-w^2} \frac{2}{-iw} e^{-iwx} \Big|_{-1}^{1}$$

$$= \frac{2i}{w} e^{-iw} + \frac{1}{w^2} \left(3e^{-iw} + e^{iw} \right) + \frac{-2i}{w^3} \left(e^{-iw} - e^{iw} \right)$$

$$= \left(\frac{2i}{w} + \frac{3}{w^2} - \frac{2i}{w^3} \right) e^{-iw} + \left(\frac{1}{w^2} + \frac{2i}{w^3} \right) e^{iw} \quad +2$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \left\{ \left(\frac{2i}{w} + \frac{3}{w^2} - \frac{2i}{w^3} \right) e^{-iw} + \left(\frac{1}{w^2} + \frac{2i}{w^3} \right) e^{iw} \right\} \quad +1$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (x+x^2) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (x+x^2) (\cos wx - i \sin wx) dx$$

$+1$ $\quad$ $+1$

$$\int_{-1}^{1} (x+x^2) \sin wx \, dx = \int_{-1}^{1} x \sin wx \, dx = \frac{1}{-w} x \cos wx \Big|_{-1}^{1} + \frac{1}{w} \int_{-1}^{1} \cos wx \, dx$$

$$= -\frac{2}{\omega} \cos \omega + \frac{1}{\omega^2} \sin wx \Big|_{-1}^{1} = -\frac{2}{w} \cos w + \frac{2}{w^2} \sin w \quad +3$$

$$\int_{-1}^{1} (x+x^2) \cos wx \, dx = \int_{-1}^{1} x^2 \cos wx \, dx = \frac{1}{\omega} x^2 \sin wx \Big|_{-1}^{1} - \frac{1}{w} \int_{-1}^{1} 2x \sin wx \, dx$$

$$= \frac{2}{\omega} \sin w - \frac{2}{w} \left[-\frac{2}{\omega} \cos w + \frac{2}{w^2} \sin w \right] = \frac{2}{\omega} \sin w + \frac{4}{\omega^2} \cos w - \frac{4}{\omega^3} \sin w$$

$+3$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \left\{ \left(\frac{2}{\omega} \sin w + \frac{4}{w^2} \cos w - \frac{4}{\omega^3} \sin w \right) + i \left(\frac{2}{\omega} \cos w - \frac{2}{w^2} \sin w \right) \right\}$$