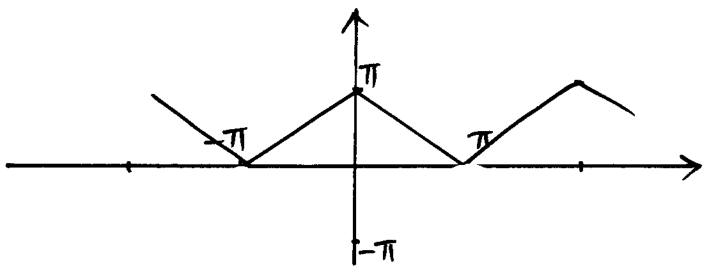
Final Exam: June 13, 2002

1. (20 points) Find the Fourier cosine series as well as the Fourier sine series of the following function. Sketch $f(x)$ and its two periodic extensions.

$$f(x) = \pi - x, \quad 0 < x < \pi.$$

① Even Extension:



+3

$$a_0 = \frac{1}{\pi}\int_0^\pi (\pi - x)\,dx = \frac{1}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^\pi = \frac{1}{2}\pi$$ +2

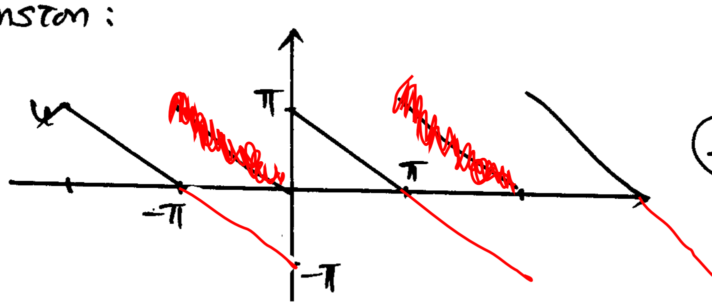
$$a_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}(\pi - x)\sin nx\right]_0^\pi + \frac{2}{\pi}\cdot\frac{1}{n}\int_0^\pi \sin nx\,dx$$ +3

$$= -\frac{2}{n^2\pi}\left[\cos nx\right]_0^\pi = \frac{2}{n^2\pi}\left[1 - \cos n\pi\right] = \frac{2(1-(-1)^n)}{n^2\pi}$$

$$f(x) = \frac{1}{2}\pi + \frac{4}{\pi}\left[\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right]$$ +2

② Odd Extension:



+3

$$b_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}(\pi - x)\cos nx\right]_0^\pi - \frac{2}{\pi}\cdot\frac{1}{n}\int_0^\pi \cos nx\,dx$$ +4

$$= \frac{2}{n} - \frac{2}{n\pi}\left[\frac{1}{n}\sin nx\right]_0^\pi = \frac{2}{n}$$

$$f(x) = 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$ +3

2. (15 points) Show that the given integral represents the indicated function:

$$\int_0^\infty \frac{\cos xw}{1+w^2}dw = \frac{\pi}{2}e^{-x}, \quad \text{if } x > 0.$$

Consider an even extension of $f(x) = \frac{\pi}{2}e^{-x}$

$$\Rightarrow A(w) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2} e^{-v} \cos wv \, dv \leftarrow +5$$

$$= \int_0^\infty e^{-v} \cos wv \, dv$$

$$= \left[\frac{1}{w} e^{-v} \sin wv\right]_0^\infty + \frac{1}{w}\int_0^\infty e^{-v} \sin wv \, dv$$

$$= \frac{1}{w}\int_0^\infty e^{-v} \sin wv \, dv$$

$$= \frac{1}{w}\left[-\frac{1}{w} e^{-v} \cos wv\right]_0^\infty - \frac{1}{w^2}\int_0^\infty e^{-v} \cos wv \, dv$$

$$= \frac{1}{w^2} - \frac{1}{w^2} A(w)$$

$$\therefore A(w) = \frac{1}{1+w^2}$$

+5

$$\Rightarrow \frac{\pi}{2}e^{-x} = \int_0^\infty A(w) \cos wx \, dw$$

$$= \int_0^\infty \frac{\cos wx}{1+w^2} dw$$

+5

3. (15 points)

(a) (5 points) Compute $\mathcal{F}_c(\frac{1}{1+x^2})$ using the result of the previous problem.

(b) (10 points) Compute $\mathcal{F}_s(xe^{-x^2/2})$ using the relation $\mathcal{F}_c(e^{-x^2/2}) = e^{-w^2/2}$.

(a) $\mathcal{F}_c\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos wx}{1+x^2} dx$

$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} e^{-w}$ +5

$= \sqrt{\frac{\pi}{2}} e^{-w}$

(b) $\mathcal{F}_s\left(x e^{-\frac{x^2}{2}}\right) = \mathcal{F}_s\left(-\left(e^{-\frac{x^2}{2}}\right)'\right)$ +4

$= -\mathcal{F}_s\left(\left(e^{-\frac{x^2}{2}}\right)'\right)$ +2

$= -\left(-w \mathcal{F}_c\left(e^{-\frac{x^2}{2}}\right)\right)$ +4

$= w \cdot e^{-\frac{w^2}{2}}$