Final Exam: June 7, 2003

1. (15 points) Assume that $f(x)$ is an odd function with the following Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x, \quad \text{for } 0 \le x \le L.$$

Show that

$$\frac{2}{L} \int_0^L [f(x)]^2 \, dx = \sum_{n=1}^{\infty} b_n^2.$$

$$\int_0^L [f(x)]^2 dx = \int_0^L \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m b_n \sin \frac{m\pi}{L} x \sin \frac{n\pi}{L} x \right) dx$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m b_n \int_0^L \left(\sin \frac{m\pi}{L} x \sin \frac{n\pi}{L} x \right) dx$$

+3

$$\int_0^L \sin \frac{m\pi}{L} x \sin \frac{n\pi}{L} x \, dx = \frac{1}{2} \int_0^L \left[\cos \frac{(m-n)\pi}{L} x - \cos \frac{(m+n)\pi}{L} x \right] dx$$

$$= \begin{cases} 0 & \text{if } m \neq n \quad +3 \\ L/2 & \text{if } m = n \quad +5 \end{cases}$$

+8

$$\therefore \frac{2}{L} \int_0^L [f(x)]^2 dx = \frac{2}{L} \sum_{n=1}^{\infty} b_n^2 \cdot \frac{L}{2}$$

$$= \sum_{n=1}^{\infty} b_n^2$$

+4

2. (25 points) Find the Fourier series of the following periodic function:

$$f(x+2\pi) = f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \sin x & \text{if } 0 \le x < \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\pi} \sin x dx = \frac{1}{2\pi}\Big[-\cos x\Big]_0^{\pi} = \frac{1}{\pi} \quad +3$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cdot \cos nx\, dx = \frac{1}{2\pi}\int_0^{\pi}\Big[\sin(n+1)x - \sin(n-1)x\Big]dx \quad +3$$

$$a_1 = \frac{1}{2\pi}\int_0^{\pi}\Big[\sin 2x\Big]dx = \frac{1}{2\pi}\Big[-\frac{1}{2}\cos 2x\Big]_0^{\pi} = 0 \quad +3$$

For $n \geq 2$,

$$a_n = \frac{1}{2\pi}\Big[-\frac{1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\Big]_0^{\pi}$$

$$= \frac{1}{2\pi}\Big[\Big(\frac{1}{n+1} - \frac{1}{n-1}\Big)\big(1-\cos(n+1)\pi\big)\Big]$$

$$= \begin{cases} \frac{1}{\pi}\cdot\frac{-2}{(n+1)(n-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad +5$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cdot \sin nx\, dx = \frac{1}{2\pi}\int_0^{\pi}\Big[\cos(n-1)x - \cos(n+1)x\Big]dx \quad +3$$

$$b_1 = \frac{1}{2\pi}\int_0^{\pi}\Big[1-\cos 2x\Big]dx = \frac{1}{2} \quad +3$$

For $n \geq 2$,

$$b_n = \frac{1}{2\pi}\Big[\frac{1}{n-1}\sin(n-1)x - \frac{1}{n+1}\sin(n+1)x\Big]_0^{\pi} = 0 \quad +3$$

$$\therefore f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{m=1}^{\infty}\frac{-2}{\pi(2m-1)(2m+1)}\cos(2mx)$$

+2

3. (20 points) Using the Fourier series of the following periodic function:

$$f(x+2\pi) = f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \sin x & \text{if } 0 \le x < \pi, \end{cases}$$

show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{m=1}^{\infty} \frac{-2}{\pi(2m-1)(2m+1)} \cos 2mx \quad +2$$

$$1 = f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} + \sum_{m=1}^{\infty} \frac{-2}{\pi(2m-1)(2m+1)} \cdot \cos m\pi$$

+10

$$\frac{1}{2} = \frac{1}{\pi} + \sum_{m=1}^{\infty} \frac{-2}{\pi(2m-1)(2m+1)} \cdot (-1)^m \quad +4$$

$$\frac{\pi}{4} = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{-1}{(2m-1)(2m+1)} \cdot (-1)^m \quad +4$$

$$= \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

4. (10 points) Find the Fourier transform of the following function

$$f(x) = \begin{cases} xe^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-iwx} \, dx \quad +1$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x e^{-x} e^{-iwx} \, dx \quad +1$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x e^{-(1+iw)x} \, dx \quad +2$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{1+iw} x e^{-(1+iw)x} \Big|_0^{\infty} + \frac{1}{1+iw} \int_0^{\infty} e^{-(1+iw)x} \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1+iw)} \int_0^{\infty} e^{-(1+iw)x} \, dx \quad +2 \quad +2$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1+iw)} \cdot \left[\frac{-1}{1+iw} e^{-(1+iw)x} \right]_0^{\infty} \quad +1$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1+iw)^2} \quad +1$$

Quiz #5 (CSE 400.001)

Tuesday, June 3, 2003

Name: ____________________ **E-mail:** ____________________

Dept: ____________________ **ID No:** ____________________

1. (10 points) Show that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}=\frac{\pi}{4}$$

using the Fourier series of the function $f(x) = 1$ $(-\pi/2 < x < \pi/2)$, $f(x) = 0$ $(\pi/2 < x < 3\pi/2)$, and $f(x) = f(x + 2\pi)$.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi/2} 1\,dx = \frac{1}{2} \qquad +1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}\sin nx\right]_0^{\pi/2} = \frac{2}{n\pi}\sin\frac{n}{2}\pi \qquad +3$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n}{2}\pi\cos nx \qquad +3$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots\right]$$

$$1 = f(0) = \frac{1}{2} + \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right] \qquad +2$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1} \qquad +1$$

2. (10 points) Using the Fourier sine integral, show the following equivalence:

$$\int_0^\infty \frac{w^3 \sin xw}{w^4+4} dw = \frac{\pi}{2} e^{-x} \cos x, \quad \text{if } x > 0.$$

Let $f(x) = \frac{\pi}{2} e^{-x} \cos x$ for $x > 0$ (+2)

$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(v) \sin \omega v \, dv$$

(+3)

$$= \int_0^\infty e^{-v} \cos v \sin \omega v \, dv$$

$$= \frac{1}{2} \int_0^\infty e^{-v} \left[\sin(\omega+1)v + \sin(\omega-1)v \right] dv$$

Using $\int_0^\infty e^{-v} \sin \alpha v \, dv = \frac{\alpha}{\alpha^2+1}$

$$B(\omega) = \frac{1}{2} \left[\frac{\omega+1}{(\omega+1)^2+1} + \frac{\omega-1}{(\omega-1)^2+1} \right]$$

$$= \frac{1}{2} \cdot \frac{(\omega^2-1)(\omega-1) + \omega+1 + (\omega^2-1)(\omega+1) + \omega-1}{(\omega^2-1)^2 + (\omega+1)^2 + (\omega-1)^2 + 1}$$

(+3)

$$= \frac{1}{2} \cdot \frac{2\omega^3}{\omega^4+4} = \frac{\omega^3}{\omega^4+4}$$

$$\frac{\pi}{2} e^{-x} \cos x = \int_0^\infty \frac{\omega^3}{\omega^4+4} \cdot \sin \omega x \, d\omega \quad \text{for } x > 0$$

(+2)