Engineering Mathematics I
(Comp 400.001)

Midterm Exam II: June 5, 2001

Problem	Score
1	
2	
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Name: ______________________

ID No: ______________________

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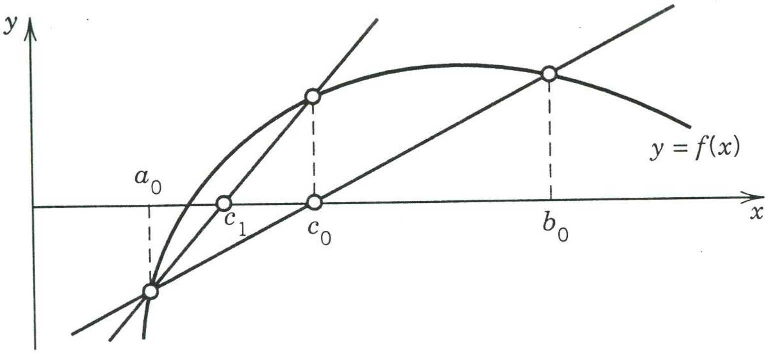

Fig. 1 Method of false position

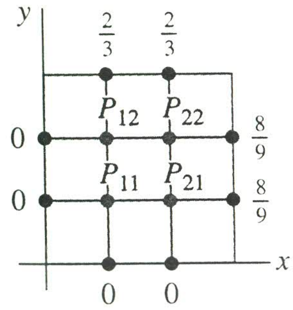

Figure 2

Table 1

Comparison of numerical methods with $h = 0.05$

x_n	Euler	Improved Euler	Runge–Kutta	True value
1.00	1.0000	1.0000	1.0000	1.0000
1.05	1.1000	1.1077		1.1079
1.10	1.2155	1.2332	1.2337	1.2337
1.15	1.3492	1.3798	1.3806	1.3806
1.20		1.5514	1.5527	1.5527
1.25	1.6849	1.7531	1.7551	1.7551
1.30	1.8955		1.9937	1.9937
1.35	2.1419	2.2721	2.2762	2.2762
1.40	2.4311	2.6060	2.6117	2.6117
1.45	2.7714	3.0038	3.0117	3.0117
1.50	3.1733	3.4795	3.4903	3.4904

Table 2

Explicit Difference Equation Approximation with $h = 0.2$, $k = 0.05$

Time	$x = 0.20$	$x = 0.40$	$x = 0.60$	$x = 0.80$
0.00	0.5878	0.9511	0.9511	0.5878
0.05	0.5597			0.5597
0.10		0.7738	0.7738	
0.15	0.3510			0.3510
0.20		0.3080	0.3080	
0.25	0.0115			0.0115
0.30	−0.1685	−0.2727	−0.2727	−0.1685
⋮	⋮			

1. (10 points) Figure 1 illustrates the basic idea for the method of False Position (Regula Falsi), which you have implemented for Homework 1. Solve the followng equation using the method of False Position:

$$f(x) = x^3 - 5x - 6 = 0.$$

Start with $a_0 = 0$ and $b_0 = 3$, and show the first three steps of computing c_0, c_1 and c_2.

$n=0$: $a_0 = 0$, $b_0 = 3$

$f(a_0) = -6$, $f(b_0) = 6$

$\therefore$ $c_0 = 1.5$, $f(c_0) = -10.125$

+2

$n=1$: $c_0 = 1.5$, $b_0 = 3$

$$c_1 = \frac{c_0 f(b_0) - b_0 f(c_0)}{f(b_0) - f(c_0)} \approx 2.442$$

+4

$f(c_1) = -3.6975$

$n=2$: $c_1 = 2.442$, $b_0 = 3$

$$c_2 = \frac{c_1 f(b_0) - b_0 f(c_1)}{f(b_0) - f(c_1)} \approx 2.6548$$

+4

$f(c_2) = \cdots$

2. (10 points) Solve the following equation by the Bisection Method:

$$f(x) = x^4 - 2 = 0.$$

Start with $a_0 = 0$ and $b_0 = 2$, and show the first three steps of computing c_0, c_1 and c_2.

$n = 0$: $a_0 = 0$, $b_0 = 2$

$f(a_0) = -2$, $f(b_0) = 14$

$c_0 = 1$, $f(c_0) = -1$

+3

$n = 1$: $c_0 = +1$, $b_0 = 2$

$c_1 = 1.5$, $f(c_1) = 3.0625$

+3

$n = 2$: $c_0 = +1$, $c_1 = 1.5$

$c_2 = 1.25$,

+4

3. (15 points) Apply the Gauss-Seidel iteration (3 steps) to the following linear system, starting from $0, 0, 0$:

$$\begin{aligned} x_1 + 9x_2 - 2x_3 &= 36 \\ 2x_1 - x_2 + 8x_3 &= 121 \\ 6x_1 + x_2 + x_3 &= 107 \end{aligned}$$

$$x_1 = \frac{1}{6}(-x_2 - x_3 + 107) = \frac{107}{6} \approx 17.8333$$

$$x_2 = \frac{1}{9}(-x_1 + 2x_3 + 36) \approx 2.018$$

$$x_3 = \frac{1}{8}(-2x_1 + x_2 + 121) \approx 10.919$$

4. (10 points) The 2×2 Hilbert matrix is

$$H_2 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

(a) Compute the condition number $\kappa(H_2)$ for the matrix norm corresponding to the l_∞ norm, and

(b) Compute the condition number $\kappa(H_2)$ for the matrix norm corresponding to the l_1 norm.

$$H_2^{-1} = 12 \cdot \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} \quad +2$$

(a) $\|H_2\|_\infty = \frac{3}{2}$ +2

$\|H_2^{-1}\|_\infty = 18$ +2

$\kappa(H_2) = 27$

(b) $\|H_2\|_1 = \frac{3}{2}$ +2

$\|H_2^{-1}\|_1 = 18$ +2

$\kappa(H_2) = 27$

5. (20 points) Table 1 compares the results of applying the Euler, improved Euler, and Runge-Kutta methods to the following initial value problem with $h = 0.05$:

$$y' = 2xy, \quad y(1) = 1.$$

Fill in the three blanks (A), (B), (C); and show your work for partial credit.

(A) Euler Method : +4

$$y_{n+1} = y_n + h f(x_n, y_n)$$
$$= y_n + 0.05 * 2 * x_n * y_n$$

(B) Improved Euler Method

k_1 : +2

k_2 : +2

$$y_{n+1} = y_n + \frac{1}{2}[k_1 + k_2] \quad +2$$

(C) Runge-Kutta Method

k_1, k_2, k_3, k_4 각 +2 × 4 = 8점

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad +2$$

6. (10 points) Set up the Gauss-Seidel Iteration that solves the following boundary value problem using the mesh shown in Figure 2:

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0, \quad 0 < x < 2,\ 0 < y < 2 \\
u(0, y) &= 0 \\
u(2, y) &= 2y - y^2 \\
u(x, 0) &= 0 \\
u(x, 2) &= \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}
\end{aligned}
$$

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = 0$$ (+2)

at P_{11}: $-4u_{11} + u_{21} + u_{12} = 0$

at P_{21}: $u_{11} - 4u_{21} + u_{22} = -\frac{8}{9}$

at P_{12}: $u_{11} - 4u_{12} + u_{22} = -\frac{2}{3}$

at P_{22}: $u_{21} + u_{12} - 4u_{22} = -\frac{14}{9}$

(+4)

$$\Rightarrow \begin{cases} u_{11} = 0.25\,u_{21} + 0.25\,u_{12} \\ u_{21} = 0.25\,u_{11} + 0.25\,u_{22} + \frac{2}{9} \\ u_{12} = 0.25\,u_{11} + 0.25\,u_{22} + \frac{1}{6} \\ u_{22} = 0.25\,u_{21} + 0.25\,u_{12} + \frac{7}{18} \end{cases}$$

(+4)

7. (25 points) Consider the following wave equation (**warning: this is different from** $u_{tt} = u_{xx}$**!**)

$$u_{tt} = 4u_{xx}, \quad 0 \le x \le 1,\ 0 \le t \le 1,$$

with boundary conditions

$$\begin{cases} u(0,t) = 0, & 0 \le t \le 1 \\ u(1,t) = 0, & 0 \le t \le 1 \\ u(x,0) = \sin \pi x, & 0 \le x \le 1 \\ u_t(x,0) = 0, & 0 \le x \le 1 \end{cases}$$

We want to solve the above equation numerically with $h = 0.2$ and $k = 0.05$. Note that $r^* \ne 1$ since $h \ne k$ and also because of the coefficient 4 in the wave equation $u_{tt} = 4u_{xx}$.

(a) (7 points) Represent $u_{i,j+1}$ in terms of $u_{i+1,j}, u_{i,j}, u_{i-1,j}, u_{i,j-1}$.

(b) (8 points) Represent $u_{i,1}$ in terms of $u_{i+1,0}, u_{i,0}, u_{i-1,0}$.

(c) (10 points) Fill in the blanks (A), (B), (C), (D), (E) in Table 2.

ⓐ $\frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = \frac{4}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$ +2

$\frac{4k^2}{h^2} = \frac{4 * 0.0025}{0.04} = 0.25$ +2

$\therefore u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = 0.25(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$

$\therefore u_{i,j+1} = 0.25(u_{i+1,j} + u_{i-1,j}) + 1.5u_{i,j} - u_{i,j-1}$ +3

ⓑ $0 = \frac{1}{2k}(u_{i,1} - u_{i,-1}) \Rightarrow u_{i,-1} = u_{i,1}$ +4

$2u_{i,1} = 0.25(u_{i+1,0} + u_{i-1,0}) + 1.5u_{i,0}$

$\therefore u_{i,1} = 0.125(u_{i+1,0} + u_{i-1,0}) + 0.75u_{i,0}$ +4

ⓒ 각 blank 당 +2점