Quiz #3

Tuesday, April 30, 2002

1. (7 points) Use $x_0 = 2$ and $x_1 = 1.75$ in solving the following equation by Newton's method

$$x^2 - 3 = 0.$$

How many iterations are necessary to produce the solution to 5D accuracy?

Solution:

$$\frac{f''(s)}{2f'(s)} \approx \frac{f''(x_1)}{2f'(x_1)} = \frac{1}{2x_1} \approx 0.29$$

$$|\epsilon_{n+1}| \approx 0.29\epsilon_n^2 \approx 0.29^3\epsilon_{n-1}^4 \approx 0.29^{2^{n+1}-1}\epsilon_0^{2^{n+1}} \leq 5\cdot 10^{-6}$$

$$\epsilon_1 - \epsilon_0 = (\epsilon_1 - s) - (\epsilon_0 - s) = -x_1 + x_0 \approx 0.25$$

$$\epsilon_1 \approx \epsilon_0 + 0.25 \approx -0.29\epsilon_0^2$$

$$0.29\epsilon_0^2 + \epsilon_0 + 0.25 \approx 0$$

$$\epsilon_0 \approx -0.27$$

$$n = 1: \quad 0.29^3 \cdot 0.27^4 \approx 8\cdot 10^{-6} > 5\cdot 10^{-6}$$

$$n = 2: \quad 0.29^7 \cdot 0.27^8 < 0.1^8 \leq 5\cdot 10^{-6}$$

Hence, $n = 2$ iterations are necessary.

2. (5 points) Interpolate

$$f_0 = f(0) = 0,\ f_1 = f(1) = 1,\ f_2 = f(2) = 6,\ f_3 = f(3) = 10$$

by the cubic spline satisfying $k_0 = 0$ and $k_3 = 0$.

Solution:

$$\begin{cases} k_0 + 4k_1 + k_2 = 3\cdot(6) = 18 \\ k_1 + 4k_2 + k_3 = 3\cdot(9) = 27 \end{cases} \Longrightarrow \begin{cases} 4k_1 + k_2 = 18 \\ k_1 + 4k_2 = 27 \end{cases} \Longrightarrow k_1 = 3, \quad k_2 = 6$$

$$\begin{cases} p_0(x) = x^3, & \text{for } 0 \leq x \leq 1 \\ p_1(x) = -(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1, & \text{for } 1 \leq x \leq 2 \\ p_2(x) = -2(x-2)^3 + 6(x-2) + 6, & \text{for } 2 \leq x \leq 3 \end{cases}$$

3. (3 points) Compute the following integral using the Gauss quadrature with $n = 3$.

$$\int_1^2 \frac{1}{1+x^2}dx$$

Solution:

$$x = \frac{1}{2}(t+3) \quad \Rightarrow \quad dx = \frac{1}{2}dt$$

$$2\int_{-1}^1 \frac{1}{t^2+6t+13}dt = 2\left[\frac{5}{9}\left(\frac{1}{3/5+6\sqrt{3/5}+13}\right) + \frac{8}{9}\cdot\frac{1}{13} + \frac{5}{9}\left(\frac{1}{3/5-6\sqrt{3/5}+13}\right)\right]$$