Engineering Mathematics I
(Comp 400.001)

Midterm Exam, October 23, 2013

< Solutions >

Problem	Score
1	
2	
3	
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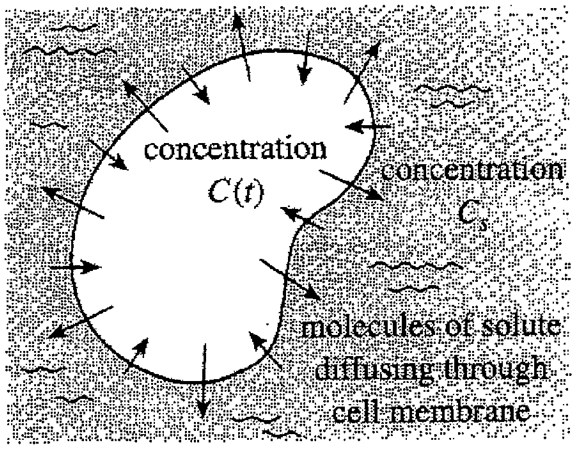
Total	

Name: ____________________

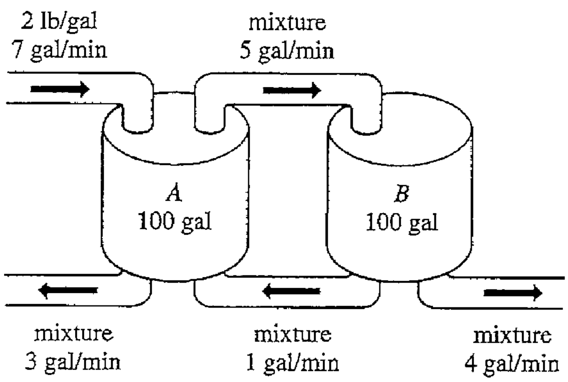
ID No: ____________________

Dept: ____________________

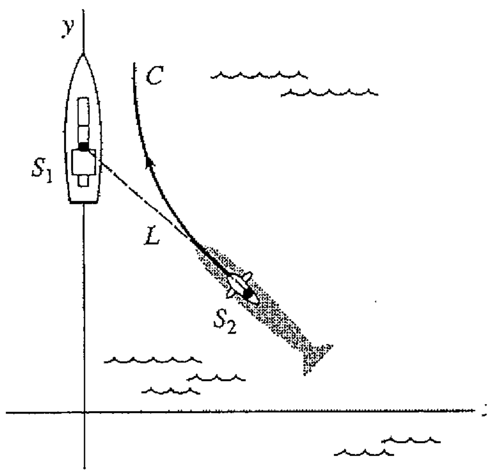
E-mail: ____________________


concentration
$C(t)$
concentration
C_s
molecules of solute
diffusing through
cell membrane

(a)


2 lb/gal
7 gal/min
mixture
5 gal/min
A
100 gal
B
100 gal
mixture
3 gal/min
mixture
1 gal/min
mixture
4 gal/min

(b)


y
C
S_1
L
S_2

(c)

1. (25 points)

(a) (10 points) Suppose a cell is suspended in a solution containing a solute of constant concentration C_s. Suppose further that the cell has constant volume V and that the area of its permeable membrane is the constant S. By Fick's law the rate of change of its mass $m(t)$ is directly proportional to the area S and the difference $C_s - C(t)$, where $C(t)$ is the concentration of the solute inside the cell at any time t. Find $C(t)$ if $m(t) = VC(t)$ and $C(0) = C_0$. See Figure (a).

(b) (5 points) Initially, two tanks A and B each hold 100 gallons of water. The water is pumped between the tanks as shown in Figure (b). Use the information given in the figure to construct a mathematical model for the numbers $X_A(t)$ and $X_B(t)$ of pounds of salt at any time in tanks A and B, respectively.

(c) (10 points) In a naval exercise, a ship S_1 is pursued by a submarine S_2, as shown in Figure (c). Ship S_1 departs point $(0,0)$ at $t = 0$ and proceeds along a straight-line course (the y-axis) at a constant speed v_1. The submarine S_2 keeps ship S_1 in visual contact, indicated by the straight dashed line L in the figure, while traveling at a constant speed v_2 along a curve C. Assume that S_2 starts at the point $(a, 0)$, $a > 0$, at $t = 0$ and that L is tangent to C. Determine a mathematical model that describes the curve C.

(a)
$$\left.\begin{aligned} m'(t) &= kS(C_s - C(t)) \\ VC'(t) &= kS(C_s - C(t)) \end{aligned}\right] \quad +3$$

$$\frac{dC}{C - C_s} = -\frac{kS}{V}dt \quad +2$$

$$C - C_s = \alpha \cdot e^{-\frac{kS}{V}t} \quad +2$$

$$C_0 - C_s = \alpha \quad +2$$

$$\therefore C(t) = C_s + (C_0 - C_s)\cdot e^{-\frac{kS}{V}t} \quad +1$$

(b)
$$x_A'(t) = -\frac{8}{100}x_A(t) + \frac{1}{100}x_B(t) + 14 \quad +3$$

$$x_B'(t) = \frac{5}{100}x_A(t) - \frac{5}{100}x_B(t) \quad +2$$

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(c)

$$\frac{dy}{dx} = \frac{v_1 t - y}{-x} = \frac{y - v_1 t}{x} \quad +3$$

$$x \cdot \frac{dy}{dx} = y - v_1 t$$

$$\cancel{\frac{dy}{dx}} + x \cdot \frac{d^2y}{dx^2} = \cancel{\frac{dy}{dx}} - v_1 \cdot \frac{dt}{dx}$$

$$x \cdot \frac{d^2y}{dx^2} + v_1 \cdot \frac{dt}{ds} \cdot \frac{ds}{dx} = 0 \quad +2$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = v_2 \quad +2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad +2$$

$$x \cdot y'' + \frac{v_1}{v_2} \cdot \sqrt{1 + (y')^2} = 0 \quad +1$$

2. (10 points) Using the method of variation of parameters,

$$y_p(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)r(x)}{W(x)}dx, \quad \text{with } W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x),$$

show that the solution to $y'' + y = r(x)$ can be written in the following form:

$$y = c_1 \cos x + c_2 \sin x + r(x) * \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$y_p(x) = -\cos x \int \sin x \cdot r(x)\,dx + \sin x \int \cos x \cdot r(x)\,dx$$

$$= -\cos x \int \sin z \cdot r(z)\,dz + \sin x \int \cos z \cdot r(z)\,dz$$

$$= \int (\sin x \cdot \cos z - \cos x \cdot \sin z)\, r(z)\,dz$$

$$= \int r(z) \cdot \sin(x - z)\,dz$$

$$y = c_1^* \cos x + c_2^* \sin x + \int r(z) \sin(x-z)\,dz$$

$$= c_1 \cos x + c_2 \sin x + \int_0^x r(z) \sin(x-z)\,dz$$

$$= c_1 \cos x + c_2 \sin x + r(x) * \sin x$$

3. (20 points) Find the general solution of the following equation

$$x^3y''' - 4x^2y'' + 8xy' - 8y = 4\ln x, \quad \text{for } x > 0.$$

Let $t = \ln x$, $x = e^t$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\cdot\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 4\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$+ 8\frac{dy}{dt} - 8y = 4t$$

$$\frac{d^3y}{dt^3} - 7\frac{d^2y}{dt^2} + 14\frac{dy}{dt} - 8y = 4t$$

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-4) = 0$$

$$\therefore y_h(t) = c_1e^t + c_2e^{2t} + c_3e^{4t}$$

$$y_p(t) = At + B \Rightarrow y_p(t) = -\frac{1}{2}t - \frac{7}{8}$$

$$\Rightarrow y(x) = c_1x + c_2x^2 + c_3x^4 - \frac{1}{2}\ln x - \frac{7}{8}$$

3. (20 points) Find the general solution of the following equation

$$x^3y''' - 4x^2y'' + 8xy' - 8y = 4\ln x, \quad \text{for } x > 0.$$

$$m(m-1)(m-2) - 4m(m-1) + 8m - 8 = 0$$

$$(m-1)(m-2)(m-4) = 0$$

$$y_h = c_1x + c_2x^2 + c_3x^4 \quad +5$$

$$y''' - \frac{4}{x}y'' + \frac{8}{x^2}y' - \frac{8}{x^3}y = \frac{4\ln x}{x^3} = r(x)$$

$$+5$$

$$y_p = \sum_{i=1}^{3} y_i \int \frac{W_i}{W} r(x)\,dx \quad +5$$

$$W = 6x^4, \quad W_1 = 2x^5, \quad W_2 = -3x^4, \quad W_3 = x^2$$

$$\therefore y_p = x\int \frac{2x^5}{6x^4}\cdot\frac{4\ln x}{x^3}\,dx + x^2\int \frac{-3x^4}{6x^4}\cdot\frac{4\ln x}{x^3}\,dx$$

$$+ x^4\int \frac{x^2}{6x^4}\cdot\frac{4\ln x}{x^3}\,dx$$

$$+5$$

$$= -\frac{1}{2}\ln x - \frac{7}{8}$$

$$\therefore y = c_1x + c_2x^2 + c_3x^4 - \frac{1}{2}\ln x - \frac{7}{8}$$

4. (20 points) Solve the following initial value problem

$$y'' + 2y' + 5y = 1 - u(t-\pi), \quad y(0) = 0, \; y'(0) = 0.$$

$$s^2Y + 2sY + 5Y = \frac{1}{s} - e^{-\pi s}\cdot\frac{1}{s} \qquad +3$$

$$Y = \left(1 - e^{-\pi s}\right)\cdot\frac{1}{s}\cdot\frac{1}{(s+1)^2+2^2} \qquad +3$$

$$= \left(1 - e^{-\pi s}\right)\cdot\frac{1}{5}\left[\frac{1}{s} - \frac{s+2}{(s+1)^2+2^2}\right]$$

$$= \frac{1-e^{-\pi s}}{5}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2}\right]$$

$$+6$$

$$y(t) = \frac{1}{5}\left[1 - \left(\cos 2t + \frac{1}{2}\sin 2t\right)\cdot e^{-t}\right] \qquad +3$$

$$- \frac{1}{5}\left[1 - \left(\cos 2(t-\pi) + \frac{1}{2}\sin 2(t-\pi)\right)\cdot e^{-t+\pi}\right]\cdot u(t-\pi)$$

$$+5$$

5. (15 points) Find the inverse Laplace transform of the following function

$$F(s) = \frac{s^{n-1}}{(s-1)^n}$$

$$\mathcal{L}[t^{n-1}] = \frac{(n-1)!}{s^n}$$

$$\mathcal{L}[e^t t^{n-1}] = \frac{(n-1)!}{(s-1)^n}$$

$$\text{Let } l_n(t) = \frac{1}{(n-1)!} e^t \cdot t^{n-1} \qquad +5$$

$$\Rightarrow l_n(0) = 0,\ l_n'(0) = 0,\ \cdots,\ l_n^{(n-2)}(0) = 0,$$

+3

$$\mathcal{L}[l_n(t)] = \frac{1}{(s-1)^n}$$

$$\mathcal{L}\left[\frac{d^{n-1}}{dt^{n-1}} l_n(t)\right] = \frac{s^{n-1}}{(s-1)^n} - s^{n-2}\cancel{l_n(0)}^{\,0} - \cdots - \cancel{l_n^{(n-2)}(0)}^{\,0}$$

+5

$$\mathcal{L}[\quad '' \quad] = \frac{s^{n-1}}{(s-1)^n}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{s^{n-1}}{(s-1)^n}\right] = \frac{d^{n-1}}{dt^{n-1}}\left[\frac{e^t \cdot t^{n-1}}{(n-1)!}\right]$$

+2

6. (10 points) Find the following integro-differential equation:

$$f'(t) - \frac{1}{2}\int_0^t (t-\tau)^2 f(\tau)\,d\tau = -t, \quad f(0) = 1.$$

$$f'(t) - \frac{1}{2} t^2 * f(t) = -t \quad +2$$

$$sF - 1 - \frac{1}{2}\cdot\frac{2!}{s^3}\cdot F = -\frac{1}{s^2} \quad +2$$

$$\left(s - \frac{1}{s^3}\right) F(s) = 1 - \frac{1}{s^2} \quad +2$$

$$F(s) = \frac{s^2-1}{s^2}\cdot\frac{s^3}{s^4-1} = \frac{s}{s^2+1} \quad +2$$

$$\therefore f(t) = \cos t \quad +2$$