

Engineering Mathematics I

Midterm Exam, October 19, 2016

1. (15 points) A young man with no initial capital invests k dollars per year at an annual rate of return r . Assume that investments are made continuously and that the return is compounded continuously.

(a) (10 points) Determine the sum $S(t)$ accumulated at any time t .

$$\begin{aligned}S'(t) &= rS(t) + k \\ \frac{dS}{dt} &= rS + k \\ \frac{dS}{S + k/r} &= r dt \\ S + k/r &= \alpha e^{rt} \\ k/r &= \alpha \\ S &= \frac{k}{r} (e^{rt} - 1)\end{aligned}$$

(b) (5 points) If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.

$$\begin{aligned}1,000,000 &= \frac{k}{0.075} (e^{0.075 \times 40} - 1) \\ k &= \frac{75,000}{e^3 - 1}\end{aligned}$$

2. (20 points) Solve the following initial value problem (without using Laplace transforms):

$$\begin{aligned} y_1' &= y_2 + 2e^t, & y_1(0) &= 1, \\ y_2' &= -y_1 + 2y_2 + 3e^t, & y_2(0) &= 1. \end{aligned}$$

Solution:

$\lambda = 1$ is a double root for the characteristic equation $-\lambda(2 - \lambda) + 1 = (\lambda - 1)^2 = 0$

$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the only eigenvector for the double root $\lambda = 1$

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}; \quad \mathbf{y}_2 = \begin{bmatrix} Ae^t + Bte^t \\ Ce^t + Dte^t \end{bmatrix}, \quad \mathbf{y}_2' = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}_2$$

$$Ae^t + Be^t + Bte^t = Ce^t + Dte^t,$$

$$Ce^t + De^t + Dte^t = (-A + 2C)e^t + (-B + 2D)te^t$$

$$A + B = C, \quad B = D \Rightarrow \text{Let } B = D = 1, \quad A = 0, \quad C = 1$$

$$\mathbf{y}_2 = \begin{bmatrix} te^t \\ (t+1)e^t \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix}, \quad \mathbf{Y}^{-1} = \begin{bmatrix} (t+1)e^{-t} & -te^{-t} \\ -e^{-t} & e^{-t} \end{bmatrix}$$

$$\mathbf{u}'(t) = \mathbf{Y}^{-1} \mathbf{g} = \begin{bmatrix} -t+2 \\ 1 \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} -\frac{1}{2}t^2 + 2t \\ t \end{bmatrix}$$

$$\mathbf{y} = c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + \left(-\frac{1}{2}t^2 + 2t\right) \mathbf{y}_1 + t \mathbf{y}_2$$

$$\mathbf{y}(0) = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c_1 = 1, \quad c_2 = 0$$

$$\mathbf{y} = \left(-\frac{1}{2}t^2 + 2t + 1\right) \begin{bmatrix} e^t \\ e^t \end{bmatrix} + \begin{bmatrix} t^2 e^t \\ (t^2 + t)e^t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}t^2 + 2t + 1\right) e^t \\ \left(\frac{1}{2}t^2 + 3t + 1\right) e^t \end{bmatrix}$$

3. (10 points) Solve the following initial value problem by the power series method. Find the recurrence formula and find the first five nonzero terms in the series.

$$y'' - 2xy' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

$$\begin{aligned} y &= \sum_{m=0}^{\infty} a_m x^m \\ y' &= \sum_{m=1}^{\infty} m a_m x^{m-1} \\ -2xy' &= \sum_{m=1}^{\infty} (-2m a_m) x^m \\ y'' &= \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = \sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s \\ a_0 &= y(0) = 1 \\ a_1 &= y'(0) = 0 \end{aligned}$$

$$y'' - 2xy' - 2y = -2a_0 + 2a_2 + \sum_{s=1}^{\infty} [(s+2)(s+1)a_{s+2} - 2sa_s - 2a_s] x^s$$

$$a_2 = a_0 = 1, \quad a_{s+2} = \frac{2}{s+2} a_s, \quad (s = 1, 2, 3, \dots)$$

$$a_{2k+2} = \frac{2}{2k+2} \cdot a_{2k} = \frac{1}{k+1} \cdot a_{2k} = \frac{1}{k+1} \cdot \frac{1}{k} \cdot a_{2(k-1)} = \frac{1}{(k+1)!}$$

$$\begin{aligned} a_{2k+1} &= \frac{2}{2k+1} \cdot a_{2k-1} = \frac{2^2}{(2k+1)(2k-1)} \cdot a_{2k-3} = \dots \\ &= \frac{2^k}{(2k+1)(2k-1) \dots 3} \cdot a_1 = 0 \end{aligned}$$

$$\begin{aligned} y &= \sum_{m=0}^{\infty} a_m x^m = \sum_{n=0}^{\infty} a_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n = e^{x^2} \\ &= 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \dots \end{aligned}$$

4. (20 points) Consider the following integral equation:

$$y(t) + 2 \int_0^t \cos(t - \tau)y(\tau)d\tau = e^{-t}. \quad (1)$$

(a) (5 points) Solve Equation (1) using the Laplace transformation.

$$\begin{aligned} y(t) + 2 \cos t * y(t) &= e^{-t} \\ Y + 2 \frac{s}{s^2 + 1} Y &= \frac{1}{s + 1} \\ \frac{(s + 1)^2}{s^2 + 1} Y &= \frac{1}{s + 1} \\ Y = \frac{s^2 + 1}{(s + 1)^3} &= \frac{1}{s + 1} - \frac{2}{(s + 1)^2} + \frac{2}{(s + 1)^3} \\ y(t) &= (1 - 2t + t^2)e^{-t} \end{aligned}$$

(b) (10 points) By differentiating Equation (1) twice, show that $y(t)$ satisfies the following initial value problem:

$$y'' + 2y' + y = 2e^{-t}, \quad y(0) = 1, \quad y'(0) = -3. \quad (2)$$

Solution:

$$\begin{aligned} y(0) + 2 \int_0^0 \cos(0 - \tau)y(\tau)d\tau &= e^0 \Rightarrow y(0) = 1. \\ y(t) + 2 \int_0^t (\cos t \cos \tau + \sin t \sin \tau)y(\tau)d\tau &= e^{-t} \\ y(t) + 2 \cos t \int_0^t \cos \tau y(\tau)d\tau + 2 \sin t \int_0^t \sin \tau y(\tau)d\tau &= e^{-t} \\ y'(t) - 2 \sin t \int_0^t \cos \tau y(\tau)d\tau + 2 \cos t \cos t y(t) + \\ 2 \cos t \int_0^t \sin \tau y(\tau)d\tau + 2 \sin t \sin t y(t) &= -e^{-t} \\ y'(t) - 2 \int_0^t \sin(t - \tau)y(\tau)d\tau + 2y(t) &= -e^{-t} \\ y'(0) + 2y(0) = -e^0 &\Rightarrow y'(0) = -3 \end{aligned}$$

$$\begin{aligned} y''(t) - 2 \int_0^t \cos(t - \tau)y(\tau)d\tau + 2y'(t) &= e^{-t} \\ y''(t) + y(t) - e^{-t} + 2y'(t) &= e^{-t} \\ y''(t) + 2y'(t) + y(t) &= 2e^{-t} \end{aligned}$$

(c) (5 points) Solve Equation (2) and verify that the solution is the same as in Equation (1).

$$\begin{aligned} \lambda^2 + 2\lambda + 1 &= 0, & (\lambda + 1)^2 &= 0 \\ y_h &= c_1 e^{-t} + c_2 t e^{-t}, & y_p &= A t^2 e^{-t} \\ y_p'' + 2y_p' + y_p &= (A t^2 - 4A t + 2A)e^{-t} + 2(-A t^2 + 2A t)e^{-t} + A t^2 e^{-t} \\ A &= 1 \\ y &= y_h + y_p = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t} \\ y(0) = c_1 &= 1, & y'(0) = -c_1 + c_2 &= -3 \Rightarrow c_2 = -2 \\ y &= e^{-t} - 2t e^{-t} + t^2 e^{-t} = (t - 1)^2 e^{-t} \end{aligned}$$

5. (15 points) Using Laplace transforms, solve the following system of differential equations

$$\begin{aligned}y_1' + y_2' + y_1 + y_2 &= 1, & y_1(0) &= 0, \\y_1' + 2y_2' + y_2 &= 0, & y_2(0) &= 1.\end{aligned}$$

Solution:

$$\begin{aligned}y_1' + 2y_1 + y_2 &= 2 \\y_2' - y_1 &= -1\end{aligned}$$

$$\begin{aligned}sY_1 + 2Y_1 + Y_2 &= \frac{2}{s} \\sY_2 - 1 - Y_1 &= -\frac{1}{s}\end{aligned}$$

$$\begin{aligned}(s+2)Y_1 + Y_2 &= \frac{2}{s} \\-Y_1 + sY_2 &= \frac{s-1}{s}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &= \frac{1}{s(s+1)^2} \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} 2 \\ s-1 \end{bmatrix} \\ &= \frac{1}{s(s+1)^2} \begin{bmatrix} s+1 \\ s^2+s \end{bmatrix} = \begin{bmatrix} \frac{1}{s(s+1)} \\ \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} \\ \frac{1}{s+1} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}y_1(t) &= 1 - e^{-t} \\y_2(t) &= e^{-t}\end{aligned}$$

6. (10 points) Find the following transformations:

(a) (3 points) $\mathcal{L}^{-1} \left[\frac{d^n}{ds^n} \frac{1}{s^2 + \omega^2} \right]$, for $n = 1, 2, 3, \dots$

Solution:

$$(-t)^n \frac{\sin \omega t}{\omega}$$

(b) (2 points) $\mathcal{L}^{-1} \left[\frac{d^n}{ds^n} \frac{s}{s^2 + \omega^2} \right]$, for $n = 1, 2, 3, \dots$

Solution:

$$(-t)^n \cos \omega t$$

(c) (5 points) $\mathcal{L} [te^{at} \sin \omega t]$

Solution:

$$-\frac{d}{ds} \left[\frac{\omega}{(s-a)^2 + \omega^2} \right] = \frac{2\omega(s-a)}{[(s-a)^2 + \omega^2]^2}$$

7. (10 points) Using Laplace transforms, solve the following initial value problem:

$$y'' + y = \delta(t - \pi)e^{2t}, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution:

$$\begin{aligned} s^2 Y - 1 + Y &= e^{-\pi(s-2)} \\ (s^2 + 1)Y &= 1 + e^{2\pi} \cdot e^{-\pi s} \\ Y(s) &= \frac{1}{s^2 + 1} + e^{2\pi} \cdot e^{-\pi s} \cdot \frac{1}{s^2 + 1} \\ y(t) &= \sin t + e^{2\pi} \cdot \sin(t - \pi)u(t - \pi) \end{aligned}$$