Quiz #6 (CSE 400.001)

Tuesday, June 1, 2004

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1. (10 points) Compute the Fourier transform of the following function:

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^1 x^2 e^{-iwx} dx \quad +2$$

$$\int_0^1 x^2 e^{-iwx} dx$$

$$= \frac{1}{-iw} x^2 e^{-iwx} \Big|_0^1 + \frac{2}{iw} \int_0^1 x e^{-iwx} dx \quad +2$$

$$= \frac{i}{w} e^{-iw} + \frac{2}{iw} \left\{ \frac{1}{-iw} x e^{-iwx} \Big|_0^1 + \frac{1}{iw} \int_0^1 e^{-iwx} dx \right\} \quad +2$$

$$= \frac{i}{w} e^{-iw} + \frac{2}{w^2} e^{-iw} - \frac{2}{w^2} \left[\frac{1}{-iw} e^{-iwx} \right]_0^1$$

$$= \frac{i}{w} e^{-iw} + \frac{2}{w^2} e^{-iw} + \frac{-2i}{w^3} \left(e^{-iw} - 1 \right) \quad +2$$

$$\therefore \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \left\{ \left(\frac{i}{w} + \frac{2}{w^2} - \frac{2i}{w^3} \right) e^{-iw} + \frac{2i}{w^3} \right\}$$

$$+2$$

2. (20 points)

(a) (8 points) Compute the Fourier series of $f(x+2\pi) = f(x) = x^2$ $(-\pi < x < \pi)$.

(b) (6 points) Show that $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - + \cdots = \frac{\pi^2}{12}$.

(c) (6 points) Show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$.

(a) $a_0 = \frac{1}{\pi}\int_0^\pi x^2 dx = \frac{\pi^2}{3}$ (+2)

$$a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{ \frac{1}{n}x^2 \cancel{\sin nx}\Big|_0^\pi - \frac{2}{n}\int_0^\pi x \sin nx\, dx \right\} \quad (+2)$$

$$= -\frac{4}{n\pi}\left\{ -\frac{1}{n}x\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\, dx \right\}$$

$$= \frac{4}{n^2}\cdot(-1)^n - \frac{4}{n^2\pi}\cdot\left[\frac{1}{n}\cancel{\sin nx}\right]_0^\pi$$

$$= \frac{4}{n^2}(-1)^n \quad (+2)$$

$$f(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \cdots \quad (+2)$$

(b) Let $x = 0$; (+3)

$$0 = \frac{\pi^2}{3} - 4 + 1 - \frac{4}{9} + \frac{1}{4} - \cdots \quad (+1)$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots \quad (+2)$$

(c) Let $x = \pi$: (+3)

$$\pi^2 = \frac{\pi^2}{3} + 4 + 1 + \frac{4}{9} + \frac{1}{4} + \cdots \quad (+1)$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \quad (+2)$$