

Chap 11. Fourier Analysis

11.1 Fourier Series

$f(x)$: periodic function $\Leftrightarrow f(x+p) = f(x)$ for all x ↗ period

$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$: periodic function of period 2π

Euler Formulas

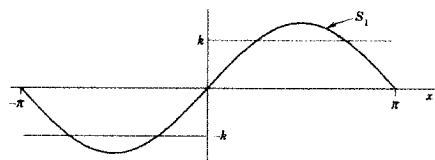
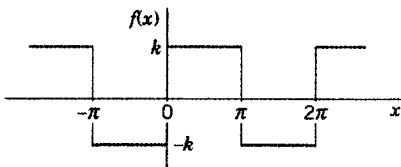
$$\begin{cases} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n=1, 2, \dots \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n=1, 2, \dots \end{cases}$$

↳ Fourier coefficients

Ex 1: $f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$ and $f(x+2\pi) = f(x)$

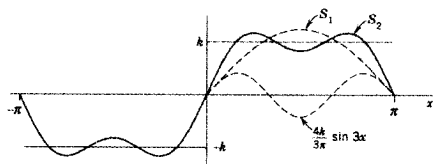
$$\Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \begin{cases} \frac{4k}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

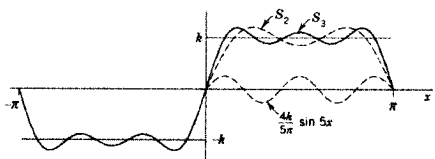


$$S_1 = \frac{4k}{\pi} \sin x$$

$$S_2 = \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x \right)$$



⋮



Orthogonality

$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$: Trigonometric system
is orthogonal

$$\langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = 0 \quad \text{if } m \neq n$$

$$\langle \sin mx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = 0 \quad \text{if } m \neq n$$

$$\langle \cos mx, \sin nx \rangle = \int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx = 0$$

11.2 Arbitrary Period ($p = 2L$)

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) \, dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi}{L}x \, dx, \quad n=1, 2, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi}{L}x \, dx, \quad n=1, 2, \dots$$

$$\text{Ex 1: } f(x) = \begin{cases} 0 & \text{if } -2 < x < -1, \\ k & \text{if } -1 < x < 1, \\ 0 & \text{if } 1 < x < 2, \end{cases} \quad \begin{array}{l} p = 2L = 4, \\ L = 2 \end{array}$$

$$\Rightarrow a_0 = \frac{1}{4} \int_{-2}^2 f(x) \, dx = \frac{1}{4} \int_{-1}^1 k \, dx = \frac{k}{2}$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi}{2}x \, dx = \frac{1}{2} \int_{-1}^1 k \cos \frac{n\pi}{2}x \, dx$$

$$= \frac{2k}{n\pi} \sin \frac{n\pi}{2} = \begin{cases} 2k/n\pi & \text{if } n=1, 5, 9, \dots \\ -2k/n\pi & \text{if } n=3, 7, 11, \dots \end{cases}$$

$$b_n = 0$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left(\cos \frac{\pi}{2}x - \frac{1}{3} \cos \frac{3\pi}{2}x + \frac{1}{5} \cos \frac{5\pi}{2}x \right. \\ \left. - \dots \right)$$

Even and Odd Functions

$$f(x) : \text{even function} \Leftrightarrow f(-x) = f(x)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x : \text{Fourier cosine series}$$

(even function of period $2L$)

$$\begin{cases} a_0 = \frac{1}{L} \int_0^L f(x) dx \\ a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n=1, 2, \dots \\ b_n = 0 \end{cases}$$

$$f(x) : \text{odd function} \Leftrightarrow f(-x) = -f(x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x : \text{Fourier sine series}$$

(odd function of period $2L$)

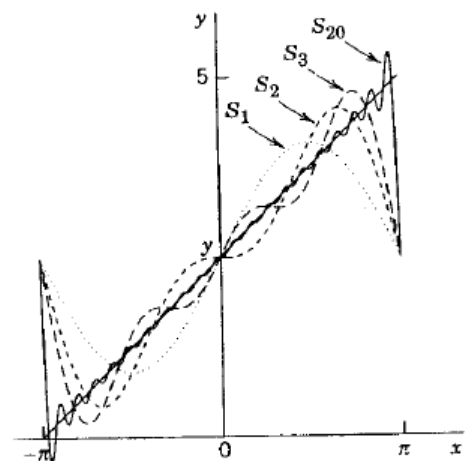
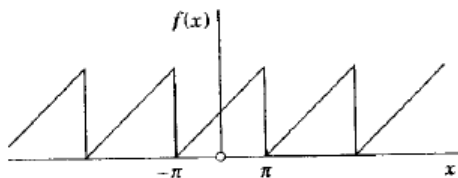
$$\begin{cases} a_n = 0, \quad n=0, 1, 2, \dots \\ b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n=1, 2, \dots \end{cases}$$

$$\text{Ex 2: } f(x) = x + \pi \quad \text{if } -\pi < x < \pi, \quad f(x+2\pi) = f(x)$$

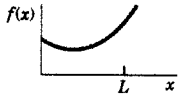
$$\Rightarrow f = f_1 + f_2, \quad \text{where } \underline{f_1(x) = x}, \quad f_2(x) = \pi$$

(odd function
with $b_n = -\frac{2}{\pi} \cos n\pi$)

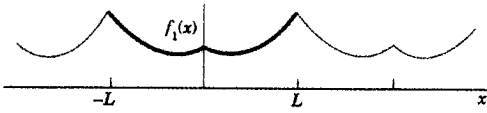
$$\Rightarrow f(x) = \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right)$$



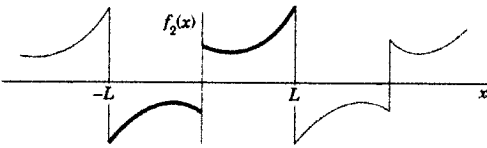
Half-Range Expansions



(0) The given function $f(x)$

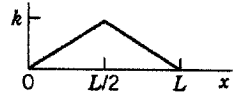


(a) $f(x)$ continued as an **even** periodic function of period $2L$



(b) $f(x)$ continued as an **odd** periodic function of period $2L$

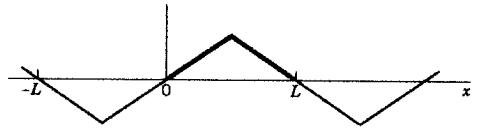
Ex 6:



$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L. \end{cases}$$



(a) Even extension



(b) Odd extension

(Solution of Ex 6)

(a) Even periodic extension

$$a_0 = \frac{1}{L} \left[\frac{2k}{L} \int_0^{L/2} x dx + \frac{2k}{L} \int_{L/2}^L (L-x) dx \right] = \frac{k}{2}$$

$$a_n = \frac{2}{L} \left[\frac{2k}{L} \int_0^{L/2} x \cos \frac{n\pi}{L} x dx + \frac{2k}{L} \int_{L/2}^L (L-x) \cos \frac{n\pi}{L} x dx \right]$$

$$= \frac{4k}{n^2 \pi^2} (2 \cos \frac{n\pi}{2} - \cos n\pi - 1)$$

$$\Rightarrow f(x) = \frac{k}{2} - \frac{16k}{\pi^2} \left(\frac{1}{2^2} \cos \frac{2\pi}{L} x + \frac{1}{6^2} \cos \frac{6\pi}{L} x + \dots \right)$$

(b) Odd periodic extension

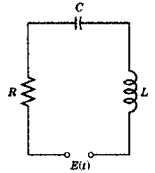
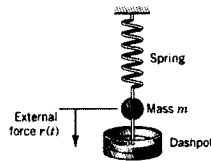
$$b_n = \frac{8k}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

$$\Rightarrow f(x) = \frac{8k}{\pi^2} \left(\frac{1}{1^2} \sin \frac{\pi}{L} x - \frac{1}{3^2} \sin \frac{3\pi}{L} x + \dots \right)$$

11.3 Forced Oscillations

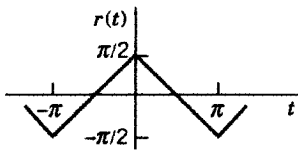
$$m y'' + c y' + k y = r(t)$$

$$L I'' + R I' + \frac{1}{C} I = \mathcal{E}(t)$$



Ex1: Let $m = 1$ (g), $c = 0.05$ (g/sec), $k = 25$ (g/sec²)

$$y'' + 0.05 y' + 25 y = r(t)$$



$$r(t) = \begin{cases} t + \frac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \frac{\pi}{2} & \text{if } 0 < t < \pi, \end{cases} \quad r(t + 2\pi) = r(t).$$

Representing $r(t)$ by a Fourier series:

$$r(t) = \frac{4}{\pi} \left(\cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \dots \right)$$

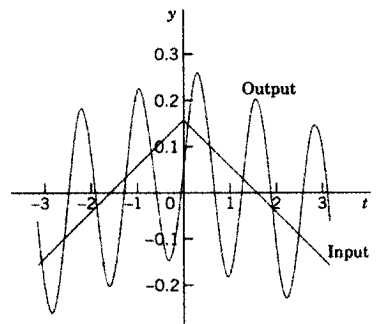
Consider $y'' + 0.05 y' + 25 y = \frac{4}{n^2 \pi} \cos nt$ ($n = 1, 3, 5, \dots$)

$$\Rightarrow y_n = A_n \cos nt + B_n \sin nt,$$

$$A_n = \frac{4(25 - n^2)}{n^2 \pi D_n}, \quad B_n = \frac{0.2}{n \pi D_n},$$

$$\text{where } D_n = (25 - n^2)^2 + (0.05n)^2$$

$$\Rightarrow y = y_1 + y_3 + y_5 + \dots$$



11.4 Approximation by Trigonometric Polynomials

$f(x)$: a periodic function of period 2π that can be represented by a Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx) : \text{Best Approximation?}$$

$\swarrow \quad \searrow$
 Fourier coefficients

Let $F(x) = A_0 + \sum_{n=1}^N (A_n \cos nx + B_n \sin nx)$

$$\begin{aligned} \Rightarrow E &= \int_{-\pi}^{\pi} (f-F)^2 dx = \int_{-\pi}^{\pi} f^2 dx - 2 \int_{-\pi}^{\pi} fF dx + \int_{-\pi}^{\pi} F^2 dx \\ &= \int_{-\pi}^{\pi} f^2 dx - 2\pi \left[2A_0 a_0 + \sum_{n=1}^N (A_n a_n + B_n b_n) \right] \\ &\quad + \pi \left[2A_0^2 + \sum_{n=1}^N (A_n^2 + B_n^2) \right] \end{aligned}$$

Take $A_n = a_n, B_n = b_n,$

Then $E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \right]$

$$E - E^* = \pi \left\{ 2(A_0 - a_0)^2 + \sum_{n=1}^N [(A_n - a_n)^2 + (B_n - b_n)^2] \right\} \geq 0$$

$\therefore E \geq E^*$

$E = E^* \Leftrightarrow A_0 = a_0, A_n = a_n, B_n = b_n, n=1, 2, \dots$

Since $E^* \geq 0,$

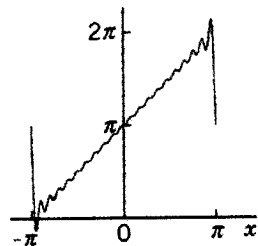
$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx : \text{Bessel Inequality}$$

Ex 1:

$f(x) = x + \pi \quad (-\pi < x < \pi)$

$$\Rightarrow F(x) = \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - + \dots + \frac{(-1)^{N+1}}{N} \sin Nx \right)$$

$$E^* = \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi \left(2\pi^2 + 4 \sum_{n=1}^N \frac{1}{n^2} \right)$$



F with $N=20$

11.7 Fourier Integral

Ex1 (Square Wave)

$$f_L(x) = \begin{cases} 0 & \text{if } -L < x < -1, \\ 1 & \text{if } -1 < x < 1, \\ 0 & \text{if } 1 < x < L, \end{cases} \quad f_L(x+2L) = f_L(x)$$

$$f(x) = \lim_{L \rightarrow \infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

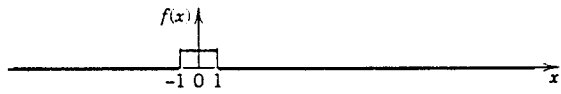
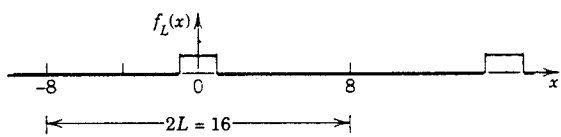
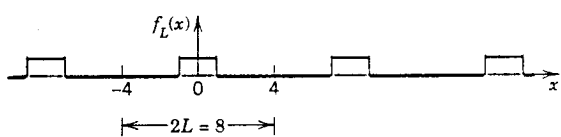
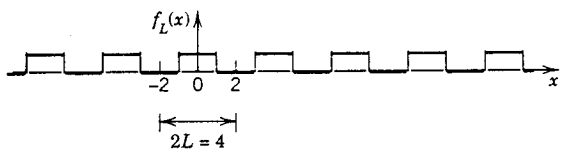
Since f_L is even, $b_n = 0$ for all n

$$a_0 = \frac{1}{2L} \int_{-L}^L dx = \frac{1}{L}$$

$$a_n = \frac{1}{L} \int_{-1}^1 \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^1 \cos \frac{n\pi x}{L} dx = \frac{2}{L} \frac{\sin(n\pi/L)}{n\pi/L}$$

↳ amplitude spectrum of f_L

Waveform $f_L(x)$



Amplitude spectrum $a_n(\omega_n)$

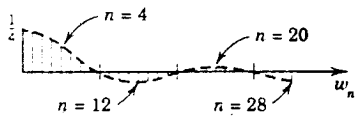
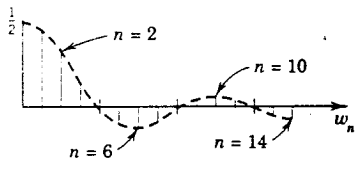
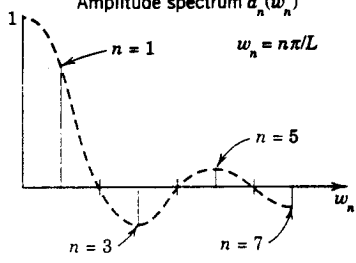


Fig. 254. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

$$f_L(x) = \frac{1}{2L} \int_{-L}^L f_L(v) dv + \frac{1}{L} \sum_{n=1}^{\infty} \left[\cos w_n x \int_{-L}^L f_L(v) \cos w_n v dv + \sin w_n x \int_{-L}^L f_L(v) \sin w_n v dv \right]$$

$$\text{Let } \Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

$$\Rightarrow \frac{1}{L} = \frac{\Delta w}{\pi}$$

$$f_L(x) = \frac{1}{2L} \int_{-L}^L f_L(v) dv + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[(\cos w_n x) \Delta w \int_{-L}^L f_L(v) \cos w_n v dv + (\sin w_n x) \Delta w \int_{-L}^L f_L(v) \sin w_n v dv \right]$$

Assuming

$$f(x) = \lim_{L \rightarrow \infty} f_L(x) : \text{absolutely integrable (i.e., } \int_{-\infty}^{\infty} |f(x)| dx < \infty)$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\cos wx \int_{-\infty}^{\infty} f(v) \cos wv dv + \sin wx \int_{-\infty}^{\infty} f(v) \sin wv dv \right] dw$$

$$= \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw : \text{Fourier Integral}$$

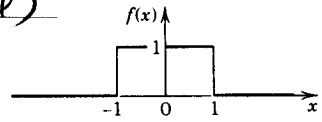
$$\text{where } A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv$$

Applications

Ex 2 (Single Pulse, Sine Integral)

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$



$$\Rightarrow A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v \, dv = \frac{2 \sin \omega}{\pi \omega}$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} 1 \cdot \sin \omega v \, dv = 0$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega x \sin \omega}{\omega} \, d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{\cos \omega x \sin \omega}{\omega} \, d\omega = \begin{cases} \pi/2 & \text{if } 0 \leq x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$\left[\int_0^{\infty} \frac{1}{2} (f(1-0) + f(1+0)) = \frac{1}{2} \right]$$

$$\text{If } x=0, \int_0^{\infty} \frac{\sin \omega}{\omega} \, d\omega = \frac{\pi}{2}$$

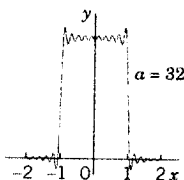
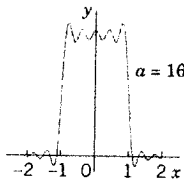
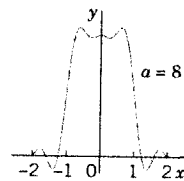
In general,

$$\int_0^a \frac{\cos \omega x \sin \omega}{\omega} \, d\omega$$

approximates

$$\int_0^{\infty} \frac{\cos \omega x \sin \omega}{\omega} \, d\omega$$

as $a \rightarrow \infty$



Fourier Cosine and Sine Integrals

If $f(x)$ is even, $B(\omega) = 0$

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos \omega x dx$$

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x d\omega$$

If $f(x)$ is odd, $A(\omega) = 0$

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin \omega x dx$$

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x d\omega$$

11. Fourier Cosine and Sine Transforms

Fourier Cosine Transforms

If $f(x)$ is even,

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x d\omega, \quad A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos \omega x dx$$

Let $\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x dx$: Fourier cosine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \cos \omega x d\omega$$

(Notation: $\mathcal{F}_c(f) = \hat{f}_c$)

Fourier Sine Transforms

If $f(x)$ is odd,

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x d\omega, \quad B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin \omega x dx$$

Let $\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x dx$: Fourier sine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(\omega) \sin \omega x d\omega$$

(Notation: $\mathcal{F}_s(f) = \hat{f}_s$)

$$\text{Ex 1: } f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

$$\Rightarrow \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} k \int_0^a \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{\sin a \omega}{\omega} \right)$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos a \omega}{\omega} \right)$$

Linearity, Transforms of Derivatives

$$(a) \mathcal{F}_c(af+bg) = a \mathcal{F}_c(f) + b \mathcal{F}_c(g)$$

$$(b) \mathcal{F}_s(af+bg) = a \mathcal{F}_s(f) + b \mathcal{F}_s(g)$$

Th 1: $f(x)$: conti. and absolutely integrable on the x -axis

$f'(x)$: piecewise conti on each finite interval

$f(x) \rightarrow 0$ as $x \rightarrow \infty$

$$\Rightarrow \textcircled{a} \mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0),$$

$$\textcircled{b} \mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f(x)\}$$

$$\Rightarrow \textcircled{a} \mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\textcircled{b} \mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}} \omega f(0)$$

Ex 3

$$f(x) = e^{-ax}, \text{ where } a > 0$$

$$\Rightarrow (e^{-ax})'' = a^2 e^{-ax}$$

$$f''(x) = a^2 f(x)$$

$$\Rightarrow a^2 \mathcal{F}_c\{f(x)\} = \mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$(a^2 + \omega^2) \mathcal{F}_c(f) = a \sqrt{\frac{2}{\pi}}$$

$$\therefore \mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + \omega^2} \right), \quad a > 0$$

11.9 Fourier Transform

Complex Form of the Fourier Integral

$$f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$$

$$\text{where } A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v dv$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(v) [\cos \omega v \cos \omega x + \sin \omega v \sin \omega x] dv d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(\omega x - \omega v) dv \right] d\omega$$

even function of ω

Note that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \sin(\omega x - \omega v) dv \right] d\omega = 0$$

odd function of ω

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) [\cos(\omega x - \omega v) + i \sin(\omega x - \omega v)] dv d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) \cdot e^{i\omega(x-v)} dv d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) \cdot e^{-i\omega v} dv \right] e^{i\omega x} d\omega$$

\Rightarrow Let

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} dx : \text{Fourier Transform}$$

$$\text{Then } f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega : \text{Inverse Fourier Transform}$$