Chap 5. Eigenvalues and Eigenvectors

5.1 Introduction (all matrices are now square.)

$\underline{A\mathbf{x} = \lambda \mathbf{x} \text{ for some } \mathbf{x} \neq \mathbf{0}}$

Ex: (Initial Value Problem for ODE)

$$\begin{cases} \dfrac{dv}{dt} = 4v - 5w, & v = 8 \text{ at } t = 0 \\ \dfrac{dw}{dt} = 2v - 3w, & w = 5 \text{ at } t = 0 \end{cases}$$

$$\mathbf{u}(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, \quad \mathbf{u}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

Matrix Form: $\dfrac{d\mathbf{u}}{dt} = A\mathbf{u}$ with $\mathbf{u} = \mathbf{u}(0)$ at $t = 0$

[Single equation: $\dfrac{du}{dt} = au$ with $u = u(0)$ at $t = 0$.
Purely exponential solution: $u(t) = e^{at} u(0)$.]

$$\Rightarrow \begin{cases} v(t) = e^{\lambda t} y \\ w(t) = e^{\lambda t} z \end{cases} \text{ or } \underline{\mathbf{u}(t) = e^{\lambda t} \mathbf{x}} \text{ in vector notation}$$

Look for pure exponential solutions:

$$\begin{cases} \lambda e^{\lambda t} y = 4 e^{\lambda t} y - 5 e^{\lambda t} z \\ \lambda e^{\lambda t} z = 2 e^{\lambda t} y - 3 e^{\lambda t} z \end{cases}$$

Cancelling $e^{\lambda t}$

Eigenvalue Problem: $\begin{cases} 4y - 5z = \lambda y \\ 2y - 3z = \lambda z \end{cases}$

Eigenvalue Equation: $\underline{A\mathbf{x} = \lambda \mathbf{x}}$ for some $\mathbf{x} \neq \mathbf{0}$

The Solutions of $A\mathbf{x} = \lambda \mathbf{x}$

$(A - \lambda I)\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$

$\boxed{5A}$ λ: eigenvalue of $A \Leftrightarrow A-\lambda I$ is singular

$\det(A-\lambda I)=0$: the characteristic equation

Each λ is associated with eigenvectors $\mathbf{x}$:

$(A-\lambda I)\mathbf{x}=\mathbf{0}$ or $A\mathbf{x}=\lambda\mathbf{x}$.

Ex: (Continued) $A-\lambda I=\begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix}$

$|A-\lambda I| = \lambda^2-\lambda-2=(\lambda+1)(\lambda-2)=0$

$\therefore \lambda=-1$ or 2

① $\lambda_1=-1$: $(A-\lambda_1 I)\mathbf{x}=\begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② $\lambda_2=2$: $(A-\lambda_2 I)\mathbf{x}=\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}_2=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Pure exponential solutions to $du/dt=Au$:

$$\left.\begin{aligned} u(t) &= e^{\lambda_1 t}\mathbf{x}_1 = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ u(t) &= e^{\lambda_2 t}\mathbf{x}_2 = e^{2t}\begin{bmatrix} 5 \\ 2 \end{bmatrix} \end{aligned}\right\}$$ special solutions

Complete Solution : $u(t)=c_1e^{\lambda_1 t}\mathbf{x}_1+c_2e^{\lambda_2 t}\mathbf{x}_2$

Initial Solution : $u(0)=c_1\mathbf{x}_1+c_2\mathbf{x}_2$ or

$$\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 8 \\ 5 \end{bmatrix} \Rightarrow c_1=3,\ c_2=1$$

The solution to the original equation:

$$u(t)=3e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}+e^{2t}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Writing the two components separately,

Solution: $\begin{cases} v(t)=3e^{-t}+5e^{2t}, & v(0)=8 \\ w(t)=3e^{-t}+2e^{2t}, & w(0)=5. \end{cases}$

Summary and Examples

Ex1: (Diagonal Matrix)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 3,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ \lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Ex2: (Projection Matrix) $\Rightarrow \lambda = 1$ or 0

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \lambda_1 = 1,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ \lambda_2 = 0,\ \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We have $\lambda = 1$ when $\mathbf{x}$ projects to itself, and $\lambda = 0$ when $\mathbf{x}$ projects to the zero vector.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ has } \lambda = 1, 1, 0, 0$$

Ex3: (Triangular Matrix)

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = (1-\lambda)\left(\tfrac{3}{4}-\lambda\right)\left(\tfrac{1}{2}-\lambda\right)$$

- The Gaussian factorization $A = LU$ is not suited to the purpose of transforming A into a diagonal or triangular matrix without changing its eigenvalues.
- The eigenvalue problem is computationally more difficult than $A\mathbf{x} = \mathbf{b}$.
- Normally, the pivots, diagonal entries, and eigenvalues are completely different. But

$\boxed{5B}$ The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\text{Trace of } A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}.$$

The product of the n eigenvalues equals the $\det(A)$.

5.2 Diagonalization of a Matrix

The eigenvectors diagonalize a matrix.

$\boxed{5C}$ Suppose the $n \times n$ matrix A has n lin. indep. eigenvectors. If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ. The eigenvalues of A are on the diagonal of Λ:

$$\text{Diagonalization} \quad \underbrace{S^{-1}AS}_{\text{the eigenvector matrix}} = \underbrace{\Lambda}_{\text{the eigenvalue matrix}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

<proof>

$$AS = A\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} Ax_1 & \cdots & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$AS = S\Lambda$, $S^{-1}AS = \Lambda$, $A = S\Lambda S^{-1}$.

(S is invertible since its columns are lin. independent) //

Remarks:

① If A has no repeated eigenvalue, then the n eigenvectors are automatically independent. (See 5D below.) Any matrix with distinct eigenvalues can be diagonalized.

② The diagonalizing matrix S is not unique.

③ Other matrices S will not produce a diagonal Λ. Suppose y is the first column of S. Then $\lambda_1 y$ is the first column of $S\Lambda$, which is Ay (the first column of AS). Then y must be an eigenvector: $Ay = \lambda_1 y$. The order of the eigenvectors of S and the eigenvalues in Λ is the same.

$\boxed{\text{Remark 4}}$ Not all matrices possess n lin. indep. eigenvectors, so not all matrices are diagonalizable. An example is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0.$$

All eigenvectors of A are multiples of $(1, 0)$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underset{\sim}{x} = \begin{bmatrix} c \\ 0 \end{bmatrix}$$

$\lambda = 0$ is a double eigenvalue — its algebraic multiplicity is 2. But the geometric multiplicity is only 1 — there is only one independent eigenvector. We can't construct S.

$\ulcorner$ ∵ Otherwise, since $\lambda_1 = \lambda_2 = 0$, Λ should be the zero matrix. But, if $\Lambda = S^{-1}AS = 0$, then $A = S\Lambda S^{-1} = 0$ # $\lrcorner$

The failure of diagonalization came from $\lambda_1 = \lambda_2$.

Ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 3$; $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1$.

These matrices are not singular. But the problem is the shortage of eigenvectors — which are needed for S.

Note:
o. Diagonalizability of A depends on enough eigenvectors.
o. Invertibility of A depends on nonzero eigenvalues.

Diagonalization can fail only if there are repeated eigenvalues. Even then, it does not always fail. $A = I$ has repeated eigenvalues $1, 1, \cdots, 1$, but it is already diagonal!

For an eigenvalue that is repeated p times, we need to check whether there are p independent eigenvectors — i.e., whether $A-\lambda I$ has rank $n-p$. To complete that circle of ideas, we have to show that distinct eigenvalues present no problem.

$\boxed{5D}$

If eigenvectors $x_1, \cdots, x_k$ correspond to different eigenvalues $\lambda_1, \cdots, \lambda_k$, then those eigenvectors are linearly independent.

$\ulcorner$ pf

Suppose $c_1 x_1 + c_2 x_2 = 0$, $\Rightarrow A(c_1 x_1 + c_2 x_2) = c_1\lambda_1 x_1 + c_2\lambda_2 x_2 = 0$.

$\Rightarrow c_1(\lambda_1 - \lambda_2)x_1 = c_1\lambda_1 x_1 + c_2\lambda_2 x_2 - \lambda_2(c_1 x_1 + c_2 x_2) = 0$

Since $\lambda_1 \neq \lambda_2$ and $x_1 \neq 0$, we have $c_1 = 0$

and similarly $c_2 = 0$.

By induction, we can extend this argument to any number of eigenvectors. $\lrcorner$

Examples of Diagonalization

Ex1: $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, AS = S\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Ex2: $K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: $90°$ rotation $\Rightarrow \det(K - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$\lambda_1 = i$: $(K - iI)x_1 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\lambda_2 = -i$: $(K + iI)x_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ and $S^{-1}KS = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Powers and Products : A^k and AB

The eigenvalues of A^2 are exactly $\lambda_1^2, \cdots, \lambda_n^2$, and every eigenvector of A is also an eigenvector of A^2.

By squaring $S^{-1}AS$, we have $S^{-1}A^2S = S^{-1}ASS^{-1}AS = \Lambda^2$.

The matrix A^2 is diagonalized by the same S, so the eigenvectors are unchanged. The eigenvalues are squared.

$\boxed{SE}$

The eigenvalues of A^k are $\lambda_1^k, \cdots, \lambda_n^k$, and each eigenvector of A is still an eigenvector of A^k. When S diagonalizes A, it diagonalizes A^k :

$$\Lambda^k = (S^{-1}AS)(S^{-1}AS)\cdots(S^{-1}AS) = S^{-1}A^kS.$$

If A is invertible, this rule also applies to its inverse ($k=-1$)

The eigenvalues of A^{-1} are $1/\lambda_i$.

「 ∵ If $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x} = \lambda A^{-1}\mathbf{x}$ and $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$. 」

Ex3 : $K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (→ 90° rotation) $\Rightarrow K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $K^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda_1 = i,\ \lambda_2 = -i$; $\lambda_1^2 = -1,\ \lambda_2^2 = -1$; $\frac{1}{\lambda_1} = -i,\ \frac{1}{\lambda_2} = i$.

$K^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Lambda^4 = \begin{bmatrix} (i)^4 & 0 \\ 0 & (-i)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

∘ For a product of two matrices, the eigenvalues of AB have no good answer. The eigenvalues of AB and $A+B$ have nothing to do with $\mu\lambda$ and $\lambda+\mu$, where λ and μ are eigenvalues of A and B, respectively

[$AB\mathbf{x} = A\mu\mathbf{x} = \mu A\mathbf{x} \neq \mu\lambda\mathbf{x}$
$(A+B)\mathbf{x} \neq (\lambda+\mu)\mathbf{x}$] since A and B may not share the same eigenvector.

Counter Example : $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 0$

But, both A and B have zero eigenvalue.

$\boxed{SH}$ Diagonalizable matrices share the same eigenvector matrix S $\Leftrightarrow$ $AB = BA$

<proof>

$\Rightarrow$) If the same S diagonalizes both $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$

$$AB = S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1\Lambda_2 S^{-1} \text{ and}$$
$$BA = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = S\Lambda_2\Lambda_1 S^{-1}.$$

Since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ (diagonal matrices always commute), we have $AB = BA$.

$\Leftarrow$) Suppose $AB = BA$. Starting from $A\mathbf{x} = \lambda\mathbf{x}$, we have

$$AB\mathbf{x} = BA\mathbf{x} = B\lambda\mathbf{x} = \lambda B\mathbf{x}.$$

Thus $\mathbf{x}$ and $B\mathbf{x}$ are both eigenvectors of A, sharing the same λ (or else $B\mathbf{x} = \mathbf{0}$).

If we assume for convenience that the eigenvalues of A are distinct – the eigenspaces are all one-dimensional – then $B\mathbf{x}$ must be a multiple of $\mathbf{x}$. In other words, $\mathbf{x}$ is an eigenvector of B as well as A.

The proof with repeated eigenvalues is a little longer. ⌟

(Spectral Theorem)

Every real symmetric matrix A can be diagonalized by an orthogonal matrix Q (i.e. $Q^TQ = I$, $Q^T = Q^{-1}$):

$$Q^{-1}AQ = \Lambda \quad \text{or} \quad A = Q\Lambda Q^T$$

The columns of Q contain orthonormal eigenvectors of A.