Chap 4. Bézier Curves

4.1 Bézier Curves

a Bézier curve of degree n is defined by

$$\mathbb{x}(t) = \mathbb{b}_0 B_0^n(t) + \mathbb{b}_1 B_1^n(t) + \cdots + \mathbb{b}_n B_n^n(t),$$

where $B_i^n(t) = \binom{n}{i}(1-t)^{n-i} t^i$: Bernstein polynomial.

$$\binom{n}{i} = \begin{cases} \frac{n!}{i!(n-i)!} & \text{if } 0 \le i \le n \\ 0 & \text{otherwise.} \end{cases}$$

4.2 Derivatives Revisited

$$\mathbb{x}'(t) = n\left[\Delta \mathbb{b}_0 B_0^{n-1}(t) + \cdots + \Delta \mathbb{b}_{n-1} B_{n-1}^{n-1}(t)\right],$$

↳ a Bezier curve of degree $(n-1)$.

where $\Delta \mathbb{b}_i = \mathbb{b}_{i+1} - \mathbb{b}_i$.

↳ vectors.

$$\frac{d^k \mathbb{x}(t)}{dt^k} = \frac{n!}{(n-k)!}\left[\Delta^k \mathbb{b}_0 B_0^{n-k}(t) + \cdots + \Delta^k \mathbb{b}_{n-k} B_{n-k}^{n-k}(t)\right],$$

where $\Delta^k \mathbb{b}_i = \Delta^{k-1}\mathbb{b}_{i+1} - \Delta^{k-1}\mathbb{b}_i$ and $\Delta^0 \mathbb{b}_i = \mathbb{b}_i$.

Ex

$$\Delta^2 \mathbb{b}_i = \Delta \mathbb{b}_{i+1} - \Delta \mathbb{b}_i = \mathbb{b}_{i+2} - 2\mathbb{b}_{i+1} + \mathbb{b}_i$$

$$\Delta^3 \mathbb{b}_i = \Delta^2 \mathbb{b}_{i+1} - \Delta^2 \mathbb{b}_i = \mathbb{b}_{i+3} - 3\mathbb{b}_{i+2} + 3\mathbb{b}_{i+1} - \mathbb{b}_i$$

$$\Delta^4 \mathbb{b}_i = \Delta^3 \mathbb{b}_{i+1} - \Delta^3 \mathbb{b}_i = \mathbb{b}_{i+4} - 4\mathbb{b}_{i+3} + 6\mathbb{b}_{i+2} - 4\mathbb{b}_{i+1} + \mathbb{b}_i$$

$$\left[\begin{array}{l} \mathbb{x}^{(k)}(0) = \frac{n!}{(n-k)!}\Delta^k \mathbb{b}_0 \\ \mathbb{x}^{(k)}(1) = \frac{n!}{(n-k)!}\Delta^k \mathbb{b}_{n-k} \end{array}\right.$$

Ex1

For a cubic Bézier curve $\mathbb{x}(t)$,

$$\mathbb{x}''(0) = 6\,\Delta^2 \mathbb{b}_0 = 6(\mathbb{b}_2 - 2\mathbb{b}_1 + \mathbb{b}_0).$$

4.3 The de Casteljau Algorithm Revisited

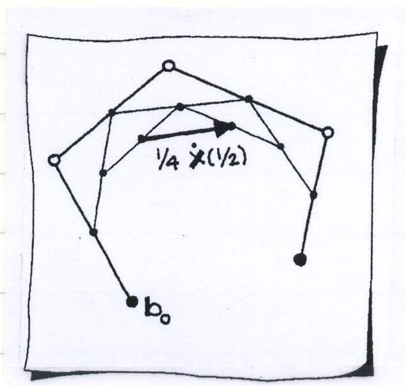
$$\left[\begin{array}{l} \text{for } r=1,\cdots,n \\ \quad \text{for } i=0,\cdots,n-r \\ \qquad \mathbf{b}_i^r(t) = (1-t)\,\mathbf{b}_i^{r-1}(t) + t\,\mathbf{b}_{i+1}^{r-1}(t) \end{array}\right.$$

- $\mathbf{x}(t) = \mathbf{b}_0^n(t)$: the point on the curve.
- The de Casteljau algorithm subdivides the curve into a left and a right segment. Their control polygons are given by

$$\mathbf{b}_0, \mathbf{b}_0^1, \cdots, \mathbf{b}_0^n \quad \text{and} \quad \mathbf{b}_0^n, \mathbf{b}_1^{n-1}, \cdots, \mathbf{b}_n.$$

- The de Casteljau algorithm provides a way for computing the first derivative and the second derivative

$$\left[\begin{array}{l} \mathbf{x}'(t) = n\left[\mathbf{b}_1^{n-1}(t) - \mathbf{b}_0^{n-1}(t)\right] \\ \mathbf{x}''(t) = n(n-1)\left[\mathbf{b}_2^{n-2}(t) - 2\mathbf{b}_1^{n-2}(t) + \mathbf{b}_0^{n-2}(t)\right] \end{array}\right.$$



4.4 The Matrix Form and Monomials Revisited

- $\mathbf{x}(t) = N^T B$, where $N = \begin{bmatrix} B_0^n(t) \\ \vdots \\ B_n^n(t) \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_0 \\ \vdots \\ \mathbf{b}_n \end{bmatrix}$
- $\mathbf{x}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \cdots + \mathbf{a}_n t^n$,

  where $\mathbf{a}_0 = \mathbf{b}_0$ and $\mathbf{a}_i = \dfrac{\frac{n!}{(n-i)!}\Delta^i \mathbf{b}_0}{i!} = \binom{n}{i}\Delta^i \mathbf{b}_0$,

  for $i=1,\cdots,n$.

4.5 Degree Elevation

o. a Bézier curve of degree n may be represented as a Bézier curve of degree $(n+1)$.

Ex

$\mathbf{x}(t) = (1-t)^2 \mathbf{b}_0 + 2(1-t)t\,\mathbf{b}_1 + t^2 \mathbf{b}_2$: a quadratic curve.

Multiplying by $1 = [(1-t)+t]$, we get

$$\mathbf{x}(t) = [(1-t)^3 + (1-t)^2 t]\,\mathbf{b}_0 + 2[(1-t)^2 t + (1-t)t^2]\,\mathbf{b}_1 + [(1-t)t^2 + t^3]\,\mathbf{b}_2$$

$$= (1-t)^3 \mathbf{b}_0 + 3(1-t)^2 t \left[\tfrac{1}{3}\mathbf{b}_0 + \tfrac{2}{3}\mathbf{b}_1\right] + 3(1-t)t^2 \left[\tfrac{2}{3}\mathbf{b}_1 + \tfrac{1}{3}\mathbf{b}_2\right] + t^3 \mathbf{b}_2 .$$

$$\therefore \mathbf{x}(t) = B_0^3(t)\,\mathbf{b}_0 + B_1^3(t)\left[\tfrac{1}{3}\mathbf{b}_0 + \tfrac{2}{3}\mathbf{b}_1\right] + B_2^3(t)\left[\tfrac{2}{3}\mathbf{b}_1 + \tfrac{1}{3}\mathbf{b}_2\right] + B_3^3(t)\,\mathbf{b}_2 .$$

Ex 4.2

$\mathbf{x}(t) = (1-t)^2 \mathbf{b}_0 + 2(1-t)t\,\mathbf{b}_1 + t^2 \mathbf{b}_2$,

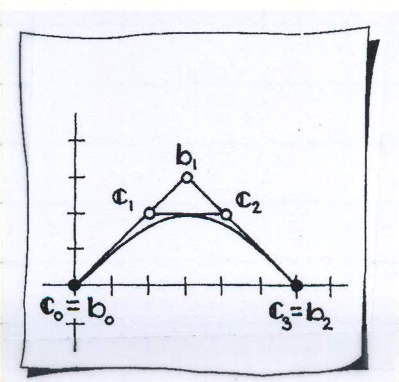
where $\mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$.

Degree elevation will produce

$$\mathbf{x}(t) = (1-t)^3 \mathbf{c}_0 + 3(1-t)^2 t\,\mathbf{c}_1 + 3(1-t)t^2 \mathbf{c}_2 + t^3 \mathbf{c}_3 ,$$

where $\mathbf{c}_0 = \mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_3 = \mathbf{b}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$\mathbf{c}_1 = \frac{1}{3}\mathbf{b}_0 + \frac{2}{3}\mathbf{b}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{c}_2 = \frac{2}{3}\mathbf{b}_1 + \frac{1}{3}\mathbf{b}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.



→ 그림 틀렸음!

Degree elevation of a Bézier curve of degree n to a Bézier curve of degree $(n+1)$:

$$\begin{cases} \mathbf{c}_0 = \mathbf{b}_0 \\ \vdots \\ \mathbf{c}_i = \frac{i}{n+1}\mathbf{b}_{i-1} + \left(1 - \frac{i}{n+1}\right)\mathbf{b}_i \\ \vdots \\ \mathbf{c}_{n+1} = \mathbf{b}_n \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & & & \\ * & * & & \\ & & \ddots & \\ & & * & * \\ & & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b}_0 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{c}_0 \\ \vdots \\ \mathbf{c}_{n+1} \end{bmatrix}$$

$(n+2) \times (n+1)$ matrix

$$\underset{\sim\sim\sim}{D\mathbf{B} = \mathbf{C}}$$

- The process of degree elevation may be repeated. The resulting sequence of control polygons converges to the curve. However, convergence is too slow.

Ex 4.3

For $n=2$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}$$

4.6 Degree Reduction

- The inverse process of degree reduction is more important. Some CAD systems allow degrees up to 40, others use cubic curves. Reducing a curve of degree 40 to cubic is non-trivial. In practice, several cubic segments will be needed, involving an interplay between subdivision and degree reduction.
- Degree reduction approximates a curve of degree $(n+1)$ by a curve of degree n. To approximate the solution of $D\mathbb{B} = \mathbb{C}$, we solve $\underline{D^T D \mathbb{B} = D^T \mathbb{C}}$.
- The matrix $D^T D$ is independent of the given data, but dependent only on n. To solve many degree reduction problems, we store the LU factorization of $D^T D$.

Ex 4.4

For $n+1 = 3$, $D^T D = \frac{1}{9}\begin{bmatrix} 10 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 10 \end{bmatrix}$

For the degree-elevated cubic curve $\mathbb{X}(t)$ with $\mathbb{c}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbb{c}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbb{c}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbb{c}_3 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, we get

$$D^T C = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 12 & 8 \\ 22 & 2 \end{bmatrix}$$

corresponds to the y-components

corresponds to the x-components

$$\Rightarrow B = (D^T D)^{-1}(D^T C) = \begin{bmatrix} 0 & 0 \\ 3 & 3 \\ 6 & 0 \end{bmatrix}$$

$\therefore \mathbb{b}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbb{b}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\mathbb{b}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$.

- In general, the degree-reduced curve may not pass through the original curve endpoint $\mathbb{c}_0$ and $\mathbb{c}_{n+1}$.

4.8 Functional Bézier Curves

- The graph of a functional curve can be thought of as a parametric curve of the form

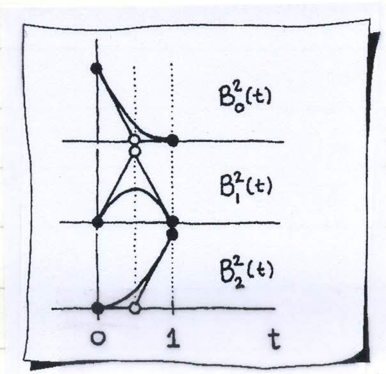
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ g(t) \end{bmatrix}. \leftarrow$$ (Another name is a nonparametric curve)

- One dimension is restricted to be a linear polynomial.
- How to write a (polynomial) functional curve in Bézier form

$$g(t) = b_0 B_0^n(t) + \cdots + b_n B_n^n(t),$$ (b_i : scalar values or Bézier ordinates)

$$\Rightarrow \mathbf{b}(t) = \begin{bmatrix} 0 \\ b_0 \end{bmatrix} B_0^n(t) + \cdots + \begin{bmatrix} j/n \\ b_j \end{bmatrix} B_j^n(t) + \cdots + \begin{bmatrix} 1 \\ b_n \end{bmatrix} B_n^n(t).$$

4.9 More on Bernstein Polynomials



Properties

- Partition of unity: $B_0^n(t) + \cdots + B_n^n(t) = 1$
- $0 \le B_i^n(t) \le 1$.

$\Rightarrow$ These two imply the convex hull property of the Bézier curve

- Convex hull property: $\mathbf{x}(t) \in CH(\mathbf{b}_0, \cdots, \mathbf{b}_n)$
- Symmetry of the Bernstein polynomials:

$$B_i^n(t) = B_{n-i}^n(1-t)$$

$\Rightarrow$ This is reflected in Bézier curves by the symmetry property.