Chap 8. Shape

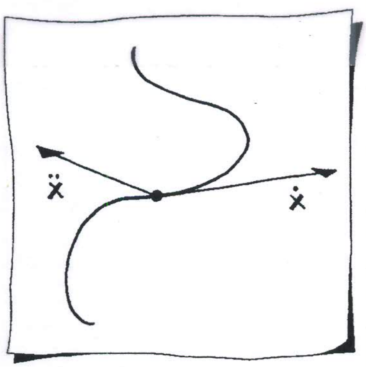
8.1 The Frenet Frame

a local coordinate system at $\mathbb{x}(t)$ with origin $\mathbb{x}(t)$ and three axes $\mathbb{t}, \mathbb{b}, \mathbb{n}$ defined by

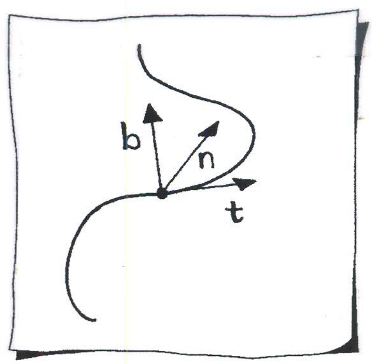
$$\mathbb{t} = \frac{\mathbb{x}'(t)}{\|\mathbb{x}'(t)\|}, \quad \mathbb{b} = \frac{\mathbb{x}'(t) \wedge \mathbb{x}''(t)}{\|\mathbb{x}'(t) \wedge \mathbb{x}''(t)\|}, \quad \mathbb{n} = \mathbb{b} \wedge \mathbb{t}.$$

↳ unit tangent ↳ binormal ↳ normal vector

- If $\mathbb{x}'(t)$ or $\mathbb{x}'(t) \wedge \mathbb{x}''(t)$ are the zero vector, then the Frenet frame is not defined.
- For planar curves, the binormal vector $\mathbb{b}(t)$ is constant.



Sketch 69.
Two derivative vectors at a point on a curve.



Sketch 70.
The Frenet frame.

8.2 Curvature and Torsion

- The rate of change of the unit vector $\mathbb{t}$ denotes the curvature of a curve. For a straight line, the curvature is zero; for a circle, it is constant.
- $\kappa(t) = \|\mathbb{x}'(t) \wedge \mathbb{x}''(t)\| / \|\mathbb{x}'(t)\|^3$.
- The osculating circle (that best approximates the curve at $\mathbb{x}(t)$) has radius $\rho = 1/\kappa$ and center $\mathbb{c}(t) = \mathbb{x}(t) + \rho(t)\,\mathbb{n}(t)$.
- The osculating circle lies in the osculating plane spanned by $\mathbb{t}$ & $\mathbb{n}$.

o. For the special case of Bézier cures, the curvatures at $t=0,1$, are given by

$$\kappa(0) = 2 \cdot \frac{n-1}{n} \frac{\text{area}[\mathbb{b}_0, \mathbb{b}_1, \mathbb{b}_2]}{\|\mathbb{b}_1 - \mathbb{b}_0\|^3}$$

$$\kappa(1) = 2 \cdot \frac{n-1}{n} \frac{\text{area}[\mathbb{b}_{n-2}, \mathbb{b}_{n-1}, \mathbb{b}_n]}{\|\mathbb{b}_n - \mathbb{b}_{n-1}\|^3}$$

o. To compute the curvature at t, $0<t<1$, just subdivide the curve at t and proceed as before.

o. A 3D curve has nonnegative curvature, by its definition. For 2D curves, we may assign a sign to the curvature by defining $\kappa(t) = \dfrac{\det[\mathbb{x}'(t)\ \mathbb{x}''(t)]}{\|\mathbb{x}'(t)\|^3}$

With this notion of curvature, we may define inflection points, where the curvature changes sign.

o. Curvature plot: the graph of $\kappa(t)$

It is a very sensitive instrument for judging a curve's shape. Wherever the curve is not "perfect", the curvature oscillates.

o. The torsion measures the change in a curve's binormal vector: $\tau(t) = \dfrac{\det[\mathbb{x}'\ \mathbb{x}''\ \mathbb{x}''']}{\|\mathbb{x}' \wedge \mathbb{x}''\|^2}$

o. The binormal of a planar curve is constant; therefore, a quadratic curve has zero torsion.

o. For the special case of Bézier curves, the torsion at $t=0$ is given by

$$\tau(0) = \frac{3}{2} \frac{n-2}{n} \frac{\text{volume}[\mathbb{b}_0, \mathbb{b}_1, \mathbb{b}_2, \mathbb{b}_3]}{\text{area}[\mathbb{b}_0, \mathbb{b}_1, \mathbb{b}_2]}.$$

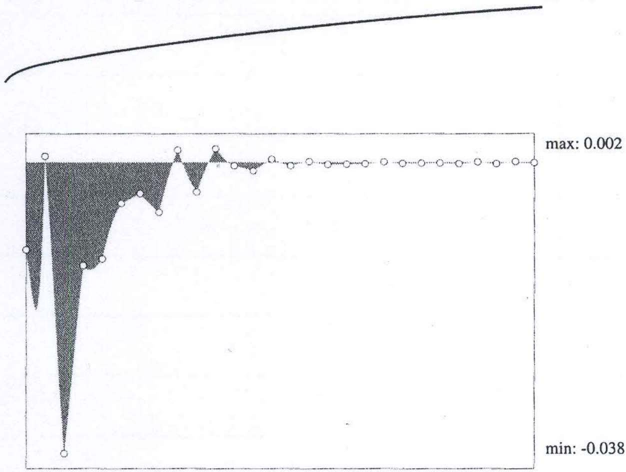


Figure 8.3.
A curve with its curvature plot.

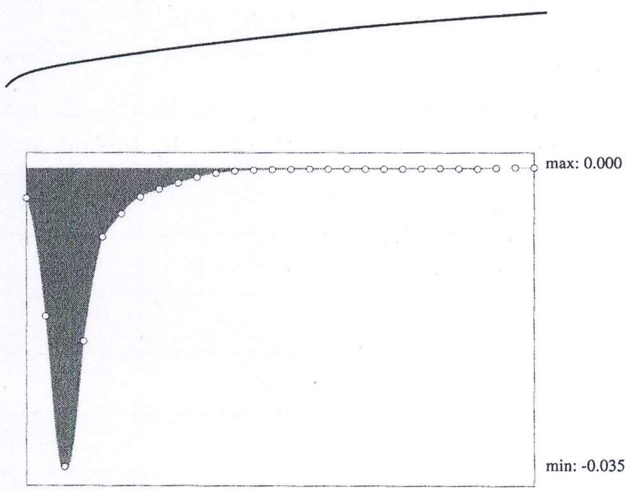
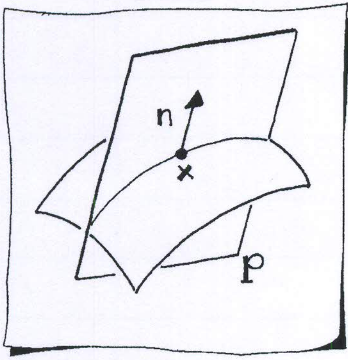


Figure 8.4.
An improved curve with its curvature plot.

8.3 Surface Curvatures

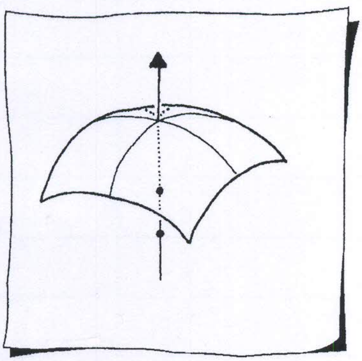
Defining the shape of surfaces is quite a bit harder than it was for curves.

We will need one basic tool: the concept of *normal curvature.* Let $\mathbf{x}(u, v)$ be a point on a surface and let $\mathbf{n}(u, v)$ be its normal. Any plane $\mathbf{P}$ through $\mathbf{x}$ which contains $\mathbf{n}$ will intersect the surface in a curve, see Sketch 75. This curve is called the *normal section* of $\mathbf{x}$ with respect to $\mathbf{P}$. It is planar by definition; we can compute its signed curvature at $\mathbf{x}$. This curvature $\kappa_{\mathbf{P}}$ is the normal curvature of the surface at point $\mathbf{x}$ with respect to the plane $\mathbf{P}$.



Sketch 75.
A normal section.

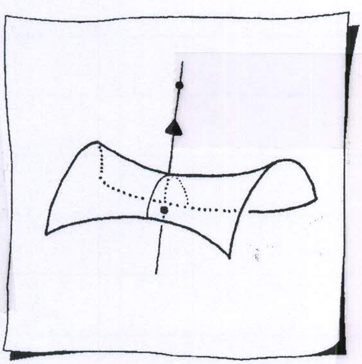
Now imagine rotating $\mathbf{P}$ around $\mathbf{n}$. For each new position of $\mathbf{P}$, we will get a new normal section, and hence a new normal curvature. Of all the normal curvatures at $\mathbf{x}$, one will be the largest, called $\kappa_{\max}$, and one will be the smallest, called $\kappa_{\min}$. These two curvatures are called the *principal curvatures* at $\mathbf{x}$. Depending on the sign of the principal curvatures, we may distinguish three cases:



Sketch 76.
An elliptic point.

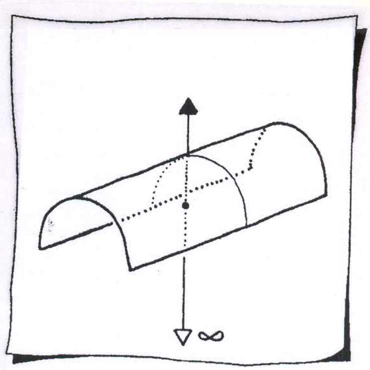
1. Both $\kappa_{\min}$ and $\kappa_{\max}$ are positive or both are negative. Then $\mathbf{x}$ is called an *elliptic point* of the surface. Sketch 76 illustrates the center of the osculating circle (8.5) for each extreme curvature. All points on a sphere or on an ellipsoid are elliptic.

2. $\kappa_{\min}$ and $\kappa_{\max}$ are of opposite sign. Then $\mathbf{x}$ is called a *hyperbolic point* of the surface. Sketch 77 illustrates. Another term is *saddle point.* All points on hyperboloids and bilinear patches are hyperbolic.[2]



Sketch 77.
A saddle point.

3. One of the principal curvatures is zero. Then $\mathbf{x}$ is called a *parabolic point.* Sketch 78 illustrates. Cylinders or cones are examples for this type.



Sketch 78.
A parabolic point.

These three cases are succinctly described by one quantity, namely the product K of $\kappa_{\min}$ and $\kappa_{\max}$:

$$K = \kappa_{\min}\kappa_{\max},$$

called *Gaussian curvature*. The sign of K determines which of the three cases best describes the shape of the surface near the point $\mathbf{x}$.

The Gaussian curvature can be computed using the first and second derivatives of the surface. We define

$$F = \det \begin{bmatrix} \mathbf{x}_u\mathbf{x}_u & \mathbf{x}_u\mathbf{x}_v \\ \mathbf{x}_u\mathbf{x}_v & \mathbf{x}_v\mathbf{x}_v \end{bmatrix}$$

and

$$S = \det \begin{bmatrix} \mathbf{n}\mathbf{x}_{u,u} & \mathbf{n}\mathbf{x}_{u,v} \\ \mathbf{n}\mathbf{x}_{u,v} & \mathbf{n}\mathbf{x}_{v,v} \end{bmatrix}.$$

All quantities involved in F and S are easily computed. The two determinants are called first and second fundamental matrices of the surface at $\mathbf{x}$.

The Gaussian curvature is then given by

$$K = \frac{S}{F}. \tag{8.9}$$

Let's revisit the list from above:

1. An elliptic point corresponds to $K > 0$.
2. A hyperbolic point corresponds to $K < 0$.
3. A parabolic point corresponds to $K = 0$.[3]

Of course most surfaces are not composed entirely of one type of Gaussian curvature.

More shape measures exist; we note two of them. The first one is *mean curvature* M. It is defined by

$$M = \frac{1}{2}[\kappa_{\min} + \kappa_{\max}].$$

It can easily be computed like this:

$$M = \frac{[\mathbf{n}\mathbf{x}_{vv}]\mathbf{x}_u^2 - 2[\mathbf{n}\mathbf{x}_{uv}][\mathbf{x}_u\mathbf{x}_v] + [\mathbf{n}\mathbf{x}_{uu}]\mathbf{x}_v^2}{F}.$$

The mean curvature is zero for surfaces that are called *minimal.* Such surfaces resemble the shape of soap bubbles.

The second curvature that is of practical use is *absolute curvature* A. It is given by

$$A = |\kappa_{\min}| + |\kappa_{\max}|$$

and measures the curvature of a surface in the most reliable way from an intuitive viewpoint.

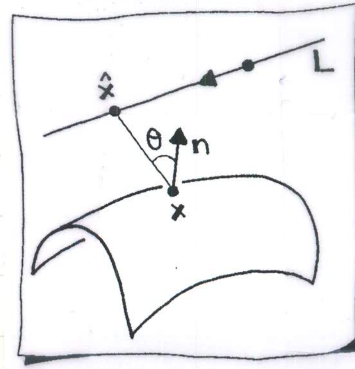
A similar expression, sometimes called the RMS (root mean square) curvature is defined as

$$R = \sqrt{\kappa_{\min}^2 + \kappa_{\max}^2}.$$

§.4 Reflection Lines

o. Highlight surface areas where reflections occur.

For any point $\mathbb{x}$ on the surface, compute its normal $\mathbb{n}$.
Let α denote the angle between $\mathbb{n}$ and the line light source $\mathbb{L}$.
If the angle α is small, the normal $\mathbb{n}$ points to the light source $\mathbb{L}$ and the corresponding surface area is highlighted.



Sketch 80.
A reflection line model.

o. To compute the angle α, we first project $\mathbb{x}$ onto $\mathbb{L}$:

$$\hat{\mathbb{x}} = \mathbb{p} + \frac{\langle \mathbb{v}, \mathbb{x} - \mathbb{p} \rangle}{\|\mathbb{v}\|^2} \mathbb{v},$$

where $\mathbb{L}$ is given by a point $\mathbb{p}$ and a vector $\mathbb{v}$.
Now α is the angle between $\mathbb{n}$ and $\hat{\mathbb{x}} - \mathbb{x}$.

o. The above method defines a curve on the given surface defined by the light line $\mathbb{L}$, referred to as an isophote.