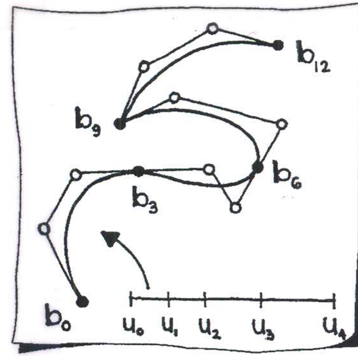
Chap 9. Composite Curves

9.1 Piecewise Bézier Curves



Sketch 81.
Three cubic Bézier curves linked together.

- A composite curve is composed of pieces; thus the term piecewise curve is also used. Each Bézier curve is then defined over an interval $[u_i, u_{i+1}]$. Let $\Delta_i = u_{i+1} - u_i$.
- Piecewise curves defined over a continuous knot sequence $(u_0 < u_1 < \cdots < u_n)$ are called spline curves. Let $\mathbf{s}$ be a spline curve.
- For the i^{th} Bézier curve $\mathbf{s}_i$, defined over $[u_i, u_{i+1}]$, we refer to its local parameter $t \in [0,1]$, $t = \frac{u - u_i}{\Delta_i}$.
- Junction points are the end points of curve segments:
  $\mathbf{s}(u_i) = \mathbf{s}_i(0) = \mathbf{s}_{i-1}(1)$.
- Derivatives of a Bézier curve:

$$\frac{d\mathbf{s}(u)}{du} = \frac{d\mathbf{s}_i(t)}{dt} \cdot \frac{dt}{du} = \frac{1}{\Delta_i} \cdot \frac{d\mathbf{s}_i(t)}{dt}.$$

- For the junction u_1 between $\mathbf{s}_0$ and $\mathbf{s}_1$,

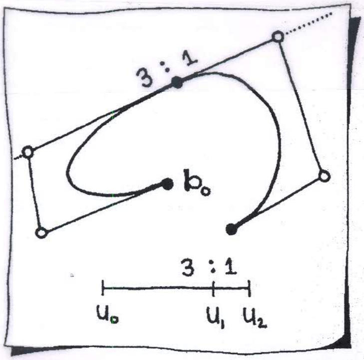
$$\mathbf{s}'(u_1^-) = \frac{3}{\Delta_0} \Delta \mathbf{b}_2 \quad \text{and} \quad \mathbf{s}'(u_1^+) = \frac{3}{\Delta_1} \Delta \mathbf{b}_3,$$

$$\mathbf{s}''(u_1^-) = \frac{6}{\Delta_0^2} \Delta^2 \mathbf{b}_1 \quad \text{and} \quad \mathbf{s}''(u_1^+) = \frac{6}{\Delta_1^2} \Delta^2 \mathbf{b}_3.$$

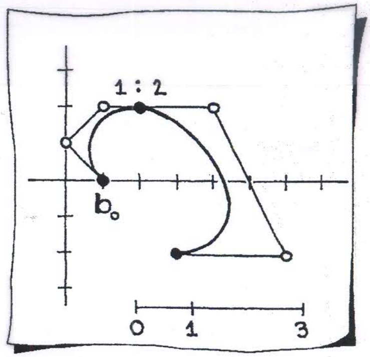
9.2 C^1 and G^1 Continuity

- To form a differentiable or C^1 curve over $[u_0, u_2]$,

$$\mathbf{b}_3 = \frac{\Delta_1}{\Delta}\mathbf{b}_2 + \frac{\Delta_0}{\Delta}\mathbf{b}_4 , \quad \text{where } \Delta = u_2 - u_0 .$$



Sketch 82.
Two C^1 cubics and their domains.



Sketch 83.
A C^1 piecewise cubic curve.

- Geometrically speaking, the three points $\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ must be collinear and they must be an affine map of the 1D parameter values u_0, u_1, u_2.

$$\text{ratio}(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4) = \frac{\Delta_0}{\Delta_1} .$$

- The concept of C^1 involves the speed of traversal of the curve, not just the curve's geometry. If we are interested purely in the shape of the curve, we need to consider Geometric Continuity, which is independent of the parameters at all. From a purly geometric viewpoint, a curve is smooth or G^1 continuous if its tangent line varies continuously. $\Rightarrow$ $\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ must be collinear.

9.3 C^2 and G^2 Continuity

- Assuming the C^1 continuity, comparing second derivatives at u_1, we get the following C^2 condition:

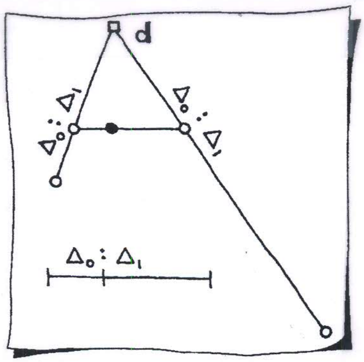
$$(\mathbf{d}_- =) -\frac{\Delta_1}{\Delta_0}\mathbf{b}_1 + \frac{\Delta}{\Delta_0}\mathbf{b}_2 = \frac{\Delta}{\Delta_1}\mathbf{b}_4 - \frac{\Delta_0}{\Delta_1}\mathbf{b}_5 \; (= \mathbf{d}_+)$$

The C^2 condition requires that $\mathbf{d}_- = \mathbf{d}_+ = \mathbf{d}$.

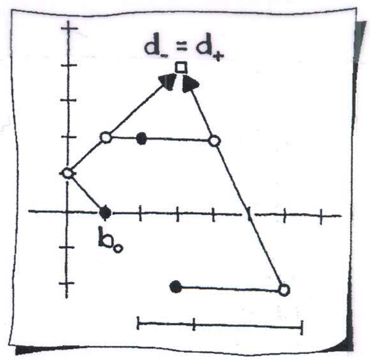
- Assuming the C^2 continuity,

$$\mathbf{b}_2 = \frac{\Delta_1}{\Delta}\mathbf{b}_1 + \frac{\Delta_0}{\Delta}\mathbf{d}, \quad \mathbf{b}_4 = \frac{\Delta_1}{\Delta}\mathbf{d} + \frac{\Delta_0}{\Delta}\mathbf{b}_5,$$

$$\mathrm{ratio}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{d}) = \mathrm{ratio}(\mathbf{d}, \mathbf{b}_4, \mathbf{b}_5) = \frac{\Delta_0}{\Delta_1}$$



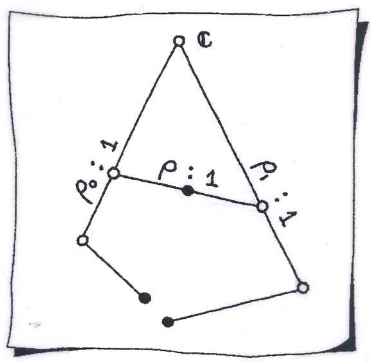
Sketch 84.
The C^2 condition.



Sketch 85.
A C^2 curve.

- If we are only interested in the geometry, the concept of curvature continuity should be used. (G^2 continuity)
- Let $\mathbf{c}$ be the intersection of two lines $\overline{\mathbf{b}_1\mathbf{b}_2}$ & $\overline{\mathbf{b}_4\mathbf{b}_5}$

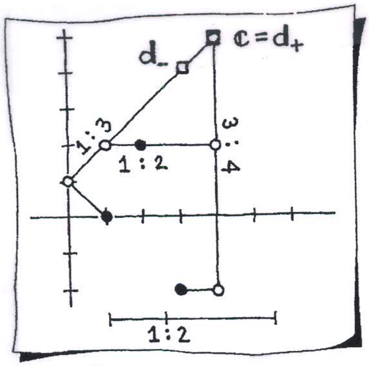
$$\rho_0 = \mathrm{ratio}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c})$$
$$\rho_1 = \mathrm{ratio}(\mathbf{c}, \mathbf{b}_4, \mathbf{b}_5)$$
$$\rho = \mathrm{ratio}(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$$



Sketch 86.
Two G^2 cubics.

- The G^2 condition: $\rho^2 = \rho_0\rho_1$.

Ex 9.2



Sketch 87.
A curve that is G^2 but not C^2.

In this example, we have

$$\rho_0 = \frac{1}{3},\ \rho_1 = \frac{3}{4},\ \rho = \frac{1}{2}$$

$$\Rightarrow \rho^2 = \rho_0\rho_1 \quad (G^2 \text{ continuous})$$

But, it is not C^2 continuous!
(since $\mathbf{d}_- \neq \mathbf{d}_+$)

9.4 Working with Piecewise Bézier Curves

- Characters are often stored as piecewise Bézier curves (outline fonts). This allows for easy rescaling. PostScript supports cubic Bézier curves.
- We pick points $\mathbf{p}_i$ on the character's outline which corresponds to significant changes in geometry. We mark tangent lines at points where the character is smooth. at corners, we mark two tangent lines. Tangent lines are represented by unit vectors $\mathbf{v}_i$. For one cubic segment, we set $\mathbf{b}_{3i} = \mathbf{p}_i$, $\mathbf{b}_{3i+3} = \mathbf{p}_{i+1}$,

$$\mathbf{b}_{3i+1} = \mathbf{b}_{3i} + 0.4\,\|\mathbf{b}_{3i+3} - \mathbf{b}_{3i}\| \cdot \mathbf{v}_i,$$

$$\mathbf{b}_{3i+2} = \mathbf{b}_{3i+3} - 0.4\,\|\mathbf{b}_{3i+3} - \mathbf{b}_{3i}\| \cdot \mathbf{v}_{i+1}.$$