

# Rational Bisectors of CSG Primitives\*

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## Abstract

The bisector surface of two rational surfaces in  $\mathbf{R}^3$  is non-rational, in general. However, in some special cases, the bisector surfaces can have rational parameterization. This paper classifies some of these special cases that are related to constructive solid geometry (CSG). We consider the bisector surfaces between points, lines, planes, spheres, cylinders, cones, and tori. Many cases are shown to yield rational bisector surfaces, while several other cases are still left as open questions.

**Key Words:** Bisector, medial surface, skeleton, CSG primitives.

## 1 Introduction

Given a solid object, its medial surface or skeleton is defined as the set of interior points which have minimum distance from at least two different points (called *foot points*) on the boundary of the solid. Medial surfaces have many important applications in engineering, in particular, for finite element mesh generation [11, 12]. (See Sherbrooke et al. [12] for a detailed survey.)

Several recent results [1, 2, 5, 6, 7, 8, 10] show that bisectors for some special pairs of rational curves and surfaces do indeed have closed-form rational representations. Considering the level of difficulty in computing the medial surface between two free-form solids, it is clear that *closed-form solutions* to some special bisector cases have significant contribution to the general skeleton problem. While the general bisector surface between two general CSG objects is highly complex, the bisectors for some special pairs of CSG primitives have closed-form rational representation. These rational bisectors need to be numerically stitched together with adjacent bisectors (which may not be rational) so as to generate the bisector

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of two CSG objects. While this is a non-trivial task that involves the computation of the intersection curves among bisector surfaces, this stitching task becomes much simpler with the introduction of closed-form rational bisectors.

Consider bisectors in  $\mathbf{R}^3$ . Depending on the locations of foot points (i.e., on the vertices, edges, and faces of the boundary), each point of the medial surface belongs to one of the following bisector surfaces: point-point, point-curve, point-surface, curve-curve, curve-surface, and surface-surface bisectors. The point-point bisector is always a plane, and hence rational. Given a point and a rational space curve, the point-curve bisector is a rational ruled surface [2]. Moreover, given two rational space curves, their bisector is a rational surface [2], except for the degenerate case where the two curves are coplanar. The bisector of a point and a rational surface is also rational [4]. However, the bisectors between a rational curve and a rational surface and between two rational surfaces are non-rational, in general.

The bisector of two algebraic surfaces is algebraic [8]. Formulated as a simultaneous system of non-linear polynomial equations, the bisector surface can be computed (or numerically approximated) based on the *dimensionality paradigm* [8]. However, it is a quite expensive computational procedure. Dutta and Hoffmann [1] consider some special cases where the bisectors of CSG primitives (i.e., plane, sphere, cylinder, cone, and torus) are given as quadrics (thus, representable as rational surfaces). Many of these cases assume special configuration of two input surfaces  $S_1$  and  $S_2$ :

1. When  $S_1$  is a plane and  $S_2$  is a torus, the plane  $S_1$  is perpendicular to the axis of  $S_2$ .
2. When  $S_1$  is a sphere and  $S_2$  is a cylinder, cone, or torus, the center of  $S_1$  is located on the axis of  $S_2$ .
3. When  $S_1$  and  $S_2$  are cylinders, cones, tori (but not simultaneously cylinders), they share the same axis.
4. When  $S_1$  and  $S_2$  are cylinders, they have the same radius or they are parallel.

In the first three cases, due to the symmetry of configuration, the bisector problem can be reduced to that of lines and circles in the plane. The planar bisectors are conic curves. By rotating these conic curves about the common axis of the two input surfaces, we can obtain quadratic surfaces of revolution. The case of two parallel cylinders can be reduced to the bisector of two circles, which is an

ellipse or hyperbola. The bisector surface is then obtained as an elliptic or hyperbolic cylinder. Finally, the case of two cylinders of equal radius is reduced to the bisector of their axes, which is a hyperbolic paraboloid.

In this paper, we add several more special cases to the results of Dutta and Hoffmann [1]. We show that the bisector surfaces are rational (but not necessarily quadric) for the following input surfaces  $S_1$  and  $S_2$ :

1. When  $S_1$  is a sphere and  $S_2$  is a CSG primitive, the bisector problem is reduced to that of the center of  $S_1$  and the offset surface of  $S_2$  (by the radius of  $S_1$ ), which produces a rational bisector surface [4].
2. When  $S_1$  and  $S_2$  are either tori or cylinders, in general position, but of equal radius, the bisector problem is reduced to that of bisectors between circles and lines in  $\mathbf{R}^3$ , which is known to yield rational bisectors [2]. Moreover, if these two reduced curves are coplanar, the bisector surface is an elliptic, hyperbolic, or parabolic cylinder and hence is rational.
3. When  $S_1$  and  $S_2$  are circular cones sharing the same apex, we show that the bisector is a rational conical surface with the same apex.
4. When  $S_1$  is a plane and  $S_2$  is either a cone or a cylinder in general position, we show that the bisector is a rational conical surface.

We first consider the bisectors of points and curves on the unit sphere  $S^2$ . This result will then be employed in constructing various rational conical bisector surfaces. Furthermore, based on the recent result of Elber and Kim [4] on the bisector of a sphere and a surface with rational offsets, we classify various special cases where the bisector surface of two CSG primitives is a rational surface. We present simple and direct methods that construct rational parameterization of these bisector surfaces.

Farouki and Johnstone [5, 6] consider the bisector problem for planar rational curves. Given a point and a rational curve in the plane, the bisector curve is rational [5]. However, in general, the bisector of two polynomial or rational curves in the plane is algebraic, but not rational [6]. Farouki and Ramamurthy [7] develop an algorithm that approximates the bisector curve using curve segments within an arbitrary error bound. (See Ramamurthy [10] for other related results.) Elber and Kim [3] transform the bisector problem for two planar rational curves  $C_1(t)$  and  $C_2(r)$  in the  $xy$ -plane into a zero-set finding problem for an algebraic equation  $\mathcal{F}(t, r) = 0$  in the  $tr$ -plane. The degree of  $\mathcal{F}(t, r)$  is significantly lower than that of the bisector curve in the  $xy$ -plane. In Section 2, we show that a similar technique can be applied to the spherical bisector problem for two rational curves  $C_1(t)$  and  $C_2(r)$  on the unit sphere.

This paper is organized as follows. In Section 2, we consider the bisector problem on the unit sphere. Section 3 presents some special cases where the bisector of two CSG primitives can be represented as a rational surface. The results of this paper are summarized in Section 4. Finally, in Section 5, we conclude the paper. The

implementation of the presented approach as well as all the figures of this paper were created with the aid of tools implemented as part of the IRIT [9] solid modeling system, developed at the Technion, Israel.

## 2 Bisectors on the Sphere

Given two points  $P$  and  $Q$  on the unit sphere  $S^2$ , their spherical (geodesic) distance  $\rho(P, Q)$  on  $S^2$  is the same as the angle between  $P$  and  $Q$ :  $\rho(P, Q) = \arccos \langle P, Q \rangle$ , where  $\langle P, Q \rangle$  is the inner product of  $P$  and  $Q$ , considered as two unit vectors. Consequently, for three points  $P, Q, R$  on the unit sphere  $S^2$ , we have  $\rho(P, Q) = \rho(P, R)$  if and only if  $\langle P, Q \rangle = \langle P, R \rangle$ .

Let  $Q$  and  $C(t)$  be a point and a rational curve, respectively, on the unit sphere  $S^2$ . Their spherical bisector curve  $\mathcal{B}(t) \in S^2$  satisfies the following three constraints (see Figure 1(a)):

$$\langle \mathcal{B}(t), Q \rangle = \langle \mathcal{B}(t), C(t) \rangle, \quad (1)$$

$$\langle \mathcal{B}(t) - C(t), C'(t) \rangle = 0, \quad (2)$$

$$\langle \mathcal{B}(t), \mathcal{B}(t) \rangle = 1. \quad (3)$$

Equation (1) constrains the bisector curve to be at equal spherical distance from  $Q$  and  $C(t)$ . Equation (2) implies that each bisector point is contained in the normal plane of  $C(t)$ . Finally, Equation (3) guarantees that the bisector curve is contained in the unit sphere. Unfortunately, Equation (3) is quadratic in  $\mathcal{B}(t)$ ; thus, it is non-trivial to solve the above three equations to get a closed-form representation for the three unknown  $x, y, z$ -coordinates of  $\mathcal{B}(t)$ .

In order to resolve the above problem, we replace Equation (3) with the following linear equation:

$$\langle \mathcal{B}(t), E_3 \rangle = 1, \quad (4)$$

where  $E_3 = (0, 0, 1)$ . That is, Equation (4) constrains the bisector curve to the plane  $Z = 1$ . Equations (1), (2), and (4) now form a set of three linear equations in three unknowns, the  $x, y, z$ -coordinates of  $\mathcal{B}(t)$ . The solution of these three equations yields a rational curve on the plane  $Z = 1$ , a curve denoted  $\overline{\mathcal{B}}(t)$ . By normalizing this curve, we get a bisector curve on the unit sphere:  $\mathcal{B}(t) = \overline{\mathcal{B}}(t) / \|\overline{\mathcal{B}}(t)\| \in S^2$ . Because of the square root function in the denominator, the bisector curve  $\mathcal{B}(t)$  is non-rational, in general.

In some applications (such as the one to be discussed in Section 3), the rational curve  $\overline{\mathcal{B}}(t)$  can be used instead of the exact bisector curve  $\mathcal{B}(t)$ . Given a rational curve,  $C(t) \in S^2$ , one can define a rational conical surface with its apex at the origin:  $S(t, v) = v * C(t)$ . Let  $L$  be a line passing through the apex (i.e., the origin). The bisector surface between  $S(t, v)$  and  $L$  can be represented as another rational conical surface (with its apex at the origin) which is defined by the rational curve  $\overline{\mathcal{B}}(t)$ . Thus, the rational curve  $\overline{\mathcal{B}}(t)$  is sufficient for this application. Figure 1(b) shows  $\mathcal{B}(t)$  and  $\overline{\mathcal{B}}(t)$  computed using the approach presented above. Figure 1(c) shows the bisector curve  $\mathcal{B}(t)$  on the unit sphere  $S^2$ . Note that if  $\mathcal{B}(t)$  is a spherical bisector, so is  $-\mathcal{B}(t)$ , i.e., the antipodal curve of  $\mathcal{B}(t)$ .

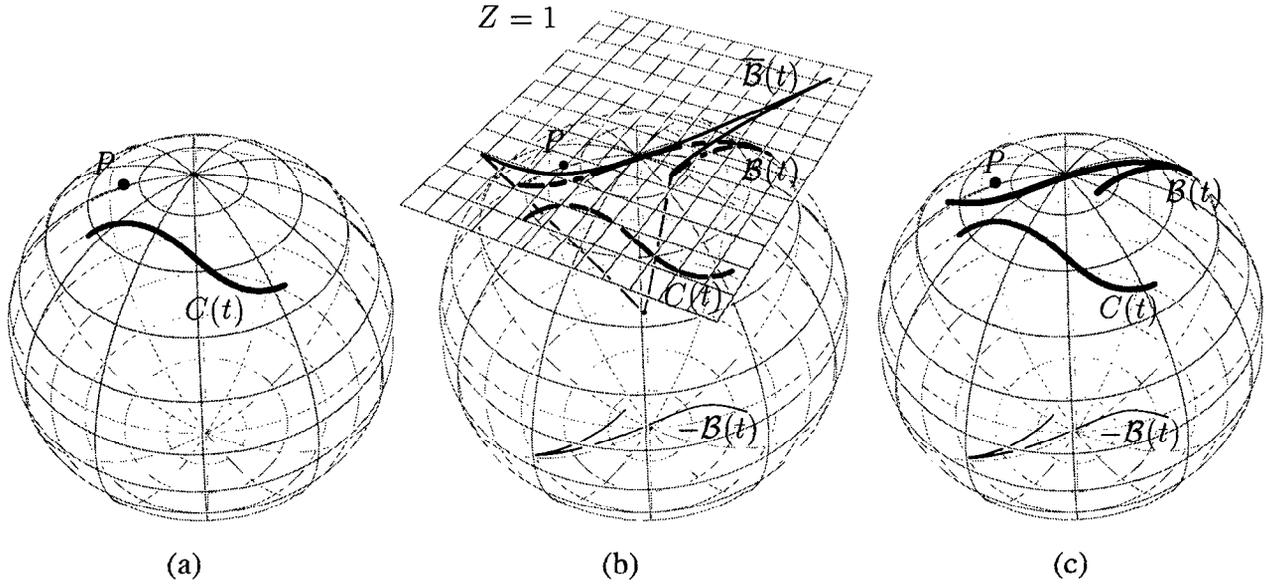


Figure 1: (a) A point  $P$  and a curve  $C(t)$  (in dark gray) on the unit sphere, (b) the rational bisector,  $\bar{B}(t)$ , on the plane  $Z = 1$  is shown in black along with its central projection onto  $S^2$ , and (c) the spherical bisector. Note that, for each bisector point, its antipodal point is also a bisector point.

Given two regular rational curves  $C_1(t)$  and  $C_2(r)$  on the sphere  $S^2$  (i.e.,  $\|C_1'(t)\| \neq 0$  and  $\|C_2'(r)\| \neq 0$ ), their bisector curve  $B(t(r)) \in S^2$  must satisfy the following three conditions in addition to the constraint of being on the unit sphere:  $\|B(t(r))\|^2 = 1$  (see Figure 2):

$$\langle B(t(r)), C_1(t) - C_2(r) \rangle = 0, \quad (5)$$

$$\langle B(t(r)) - C_1(t), C_1'(t) \rangle = 0, \quad (6)$$

$$\langle B(t(r)) - C_2(r), C_2'(r) \rangle = 0. \quad (7)$$

Equations (5)–(7) are linear in  $B(t(r))$ ; thus, there is a unique solution if and only if the three vectors  $C_1(t) - C_2(r)$ ,  $C_1'(t)$ , and  $C_2'(r)$  are linearly independent [2]. However, this is not the case when the points in curves  $C_1(t)$  and  $C_2(r)$  are foot points of a bisector point  $B(t(r))$  on the sphere  $S^2$ . Consider the set of points (on the sphere  $S^2$ ) which are at a spherical distance  $\rho_0$  from  $B(t(r))$ , where  $\rho_0 = \rho(C_1(t), B(t(r))) = \rho(C_2(r), B(t(r)))$ . This set generates a circle  $\mathcal{C}$  on the sphere  $S^2$  with its center at  $\cos(\rho_0)B(t(r)) \in \mathbb{R}^3$  and radius  $\sin(\rho_0)$ . The tangent vectors  $C_1'(t)$  and  $C_2'(r)$  are also tangent to the circle  $\mathcal{C}$ ; thus, they are contained in the plane  $\mathcal{P}$  of the circle  $\mathcal{C}$ . Moreover, the difference vector  $C_1(t) - C_2(r)$  is also contained in the same plane  $\mathcal{P}$ . This means that the three vectors  $C_1(t) - C_2(r)$ ,  $C_1'(t)$ , and  $C_2'(r)$  are coplanar and linearly dependent.

On the other hand, if the three vectors  $C_1(t) - C_2(r)$ ,  $C_1'(t)$ , and  $C_2'(r)$  are coplanar, they determine a plane  $\mathcal{P}$  and a circle  $\mathcal{C} = \mathcal{P} \cap S^2$ . It is easy to check that the radial line passing through the origin and the center of  $\mathcal{C}$  intersects the sphere,  $S^2$ , in two points, each of which is a bisector point with its foot points at  $C_1(t)$  and  $C_2(r)$ . Consequently, the following condition classifies all  $(t, r)$ -

pairs that generate bisector points:

$$\Delta(r, t) = \begin{vmatrix} C_1(t) - C_2(r) \\ C_1'(t) \\ C_2'(r) \end{vmatrix} = 0. \quad (8)$$

Since we assume that the two input curves  $C_1(t)$  and  $C_2(r)$  are regular, the above condition is a necessary and sufficient condition for the  $(t, r)$ -pair to generate a bisector point. Figure 2 shows two examples of spherical bisectors.

### 3 Bisectors of CSG Primitives

Dutta and Hoffmann [1] classify some special cases where the bisector surface of two CSG primitives can be represented as a quadric and hence is a rational surface. In this section, we add several more special cases which can be represented as rational, but not necessarily quadric, surfaces. Moreover, we present simple methods for constructing rational parameterization of these bisector surfaces.

#### 3.1 Bisectors of Points, Lines, and Planes

The bisectors of points, lines, and planes are mostly quite straightforward to compute. The bisector of two points or two planes is a plane. The bisector of two lines is a hyperbolic paraboloid if the lines are skew; a plane if the lines are parallel; and two planes if the lines intersect [1, 2]. The bisector of a point and a plane is a paraboloid of revolution [1, 4]. The bisector of a point and a line is a parabolic cylinder. It is clear that these bisector surfaces are all rational surfaces. Elber and Kim [2] present a simple method for constructing rational parameterization of these quadric bisector

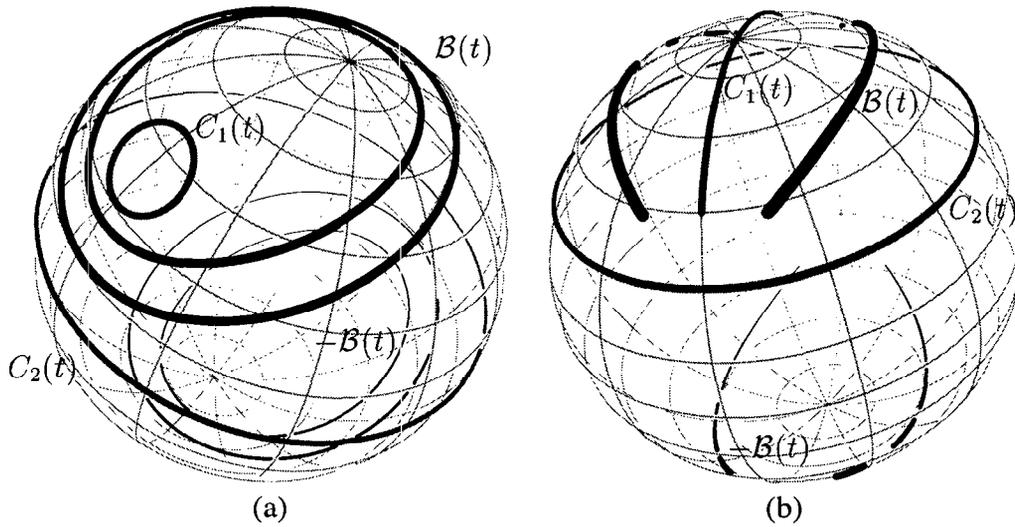


Figure 2: Two examples of spherical curve-curve bisector on  $S^2$ : (a) the bisectors of two circles,  $C_i(t)$ , on  $S^2$ , and (b) the bisectors of an arc,  $C_1(t)$ , and a circle,  $C_2(t)$ . Given arc and circles are shown in dark gray and bisectors are shown in black.

surfaces. The line-plane is the only non-trivial case left for consideration among these simple primitives. We will consider the line-plane bisector in Section 3.3 as a special case of line-cone bisector where the cone degenerates to a plane.

### 3.2 Bisector of a Cone and a Line

Consider the bisector of a cone and a line through its apex. Without loss of generality, we may assume that the apex of the cone is at the origin. Thus, this bisector problem can be formulated in terms of the radial lines emanating from the origin. The angle between two radial lines may serve as the distance measure between the two lines. The bisector surface is then a conical surface with its apex at the origin.

The bisector problem can be reduced to a similar problem on the sphere. The cone and the line are reduced to a circle and a point on the sphere,  $S^2$ , (or two circles and two points if antipodal points are considered identical), which are obtained by intersecting the cone and the line with the sphere. The spherical bisector of a point and a curve on  $S^2$  was already considered in Section 2. The bisector curve can be computed and represented as a rational curve  $\bar{B}(u)$  on the plane  $Z = 1$ . Hence, the bisector surface of a cone and a line through its apex can be represented as a rational surface:

$$B(u, v) = v * \bar{B}(u). \quad (9)$$

### 3.3 Bisector of a Plane and a Line

We may assume that the plane is the  $xy$ -plane and the line intersects the plane at the origin. The plane may be considered as a cone with its apex at the origin and its spanning angle  $\pi/2$ . Thus, the line-plane bisector is rational. Figure 3(a) shows an example of a line-plane bisector.

### 3.4 Bisectors of a Sphere and a CSG Primitive

The bisector of a point  $P \in \mathbb{R}^3$  and a rational surface  $S(u, v)$  is always rational [4]. Moreover, if  $S(u, v)$  has a rational offset surface, the bisector of a sphere and the surface  $S(u, v)$  is also rational [4]. When we offset the sphere (inward) and the surface  $S(u, v)$  by the radius of the sphere, the sphere-surface bisector problem is reduced to a point-surface bisector problem. Since all CSG primitives are closed under the offset operation, the bisector of a sphere and a CSG primitive is always rational. Figure 4 shows several examples of sphere-cone, sphere-cylinder, sphere-sphere, and sphere-plane bisectors, each computed via the reduction to a point-surface bisector problem. Figure 5 shows two examples of sphere-torus bisectors.

Other types of bisector problems can also be reduced to a point-surface bisector. The sphere-line bisector can be reduced to a point-cylinder bisector, via an offset by the radius of the sphere. Since the point-cylinder bisector is rational, the sphere-line bisector is also rational. Figure 3(b) shows an example of line-sphere bisector. The sphere-plane bisector is also rational since it can be reduced to a point-plane bisector.

### 3.5 Bisector of a Cylinder and a Plane

We can apply a similar reduction scheme to the construction of a cylinder-plane bisector. The cylinder is shrunk to its axis by offsetting the cylinder (inward) by its radius. At the same time, the plane is offset to a (parallel) plane by the same offset radius. The bisector is then constructed as a line-plane bisector which is known to be rational by Section 3.3. Figure 6(a) shows an example of a cylinder-plane bisector.

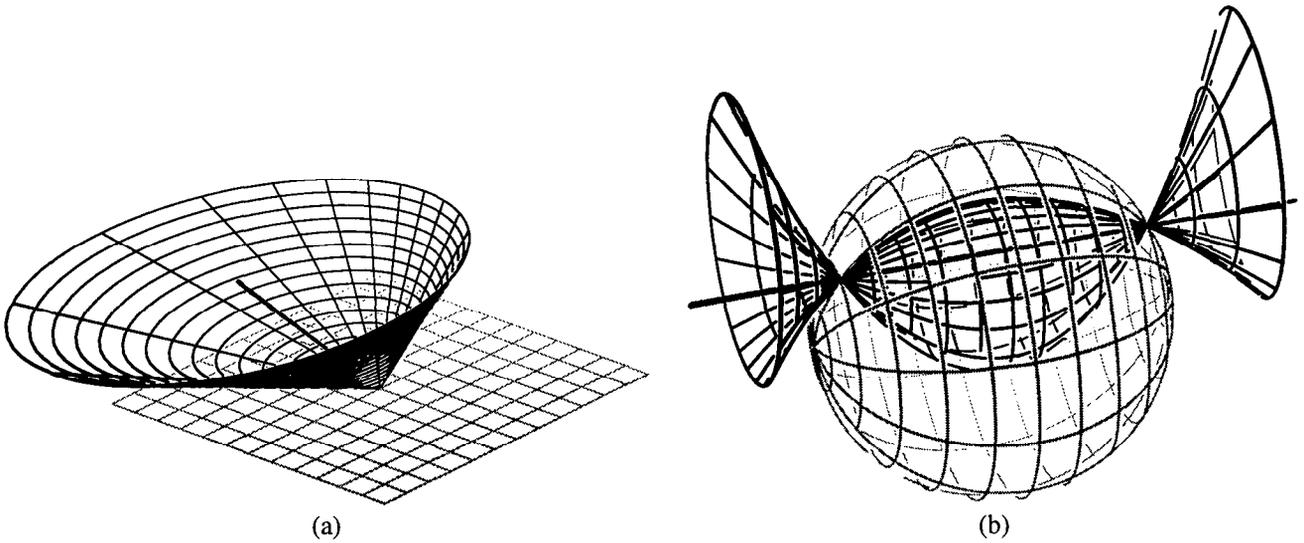


Figure 3: (a) The rational bisector (in black) of a plane (in gray) and a line (in dark gray), and (b) the rational bisector (in black) of a sphere (in gray) and a line (in dark gray).

### 3.6 Bisector of a Cone and a Plane

The bisector of a cone and a plane is rather more complex. We assume that the plane is the  $xy$ -plane. Moreover, let  $P = (p_x, p_y, p_z)$  be the apex of the cone,  $\vec{V} = (v_x, v_y, v_z)$  be the direction of the axis of the cone, and  $\alpha$  be the spanning angle of the cone. We can gradually offset both the  $xy$ -plane and the cone until the apex of an offset cone is contained in the offset plane. Let  $r$  denote the corresponding offset radius. The apex of the cone then moves along the axis of the cone (or along the direction  $\vec{V}$ ) by distance of  $(r/\sin \alpha)$ . When we measure the motion of the apex along the normal of the plane, we get a vertical distance of

$$\left\langle \frac{r}{\sin \alpha} \vec{V}, (0, 0, 1) \right\rangle = \frac{r}{\sin \alpha} v_z.$$

Thus, we have the following relation:

$$p_z = r + \frac{r \cdot v_z}{\sin \alpha}, \quad (10)$$

or

$$r = \frac{p_z}{1 + \frac{v_z}{\sin \alpha}}. \quad (11)$$

We may consider the offset plane as a circular cone with its apex at the same position as that of the offset cone. The cone-plane bisector problem, where the cone's apex is contained in the plane, can then be further reduced to the bisector problem of two cones sharing the same apex. This problem is discussed below.

### 3.7 Bisector of Two Circular Cones

We consider the bisector of two cones sharing the same apex. Applying the transformation technique of Section 2, the bisector problem is reduced to that of two circles on  $S^2$ . We further offset both circles on the sphere until the smaller one vanishes to a point. In

other words, the spherical bisector of two circles is reduced to that of a point and an offset circle, which is known to have a rational representation on the plane  $Z = 1$ . Therefore, the bisector of two circular cones sharing an apex is rational. Figure 6(b) shows an example of cone-plane bisector.

## 4 Summary of Result

Table 1 summarizes the various special cases where we can represent the bisector of two CSG primitives as a rational surface. Many of these cases can be reduced to simpler cases. The remarks below further enumerate the items in Table 1:

- (1) If the line is parallel to the cylinder, the bisector is an elliptic or a hyperbolic cylinder [1].
- (2) If the line passes through the apex of the cone, the bisector is a rational conical surface with the same apex.
- (3) If the line is coincident with the axis of the torus, the rational bisector is a surface of revolution with a parabolic cross section.
- (4) A rational surface, via a reduction to a plane-line bisector. If the line is parallel to the plane, the bisector is a parabolic cylinder. Otherwise, the bisector is an elliptic cone [1].
- (5) A paraboloid of revolution [1], via a reduction to a plane-point bisector.
- (6) A rational surface, via a reduction to a cone-line bisector.
- (7) If the plane is perpendicular to the axis of the torus, the rational bisector is a surface of revolution with a parabolic cross section [1].

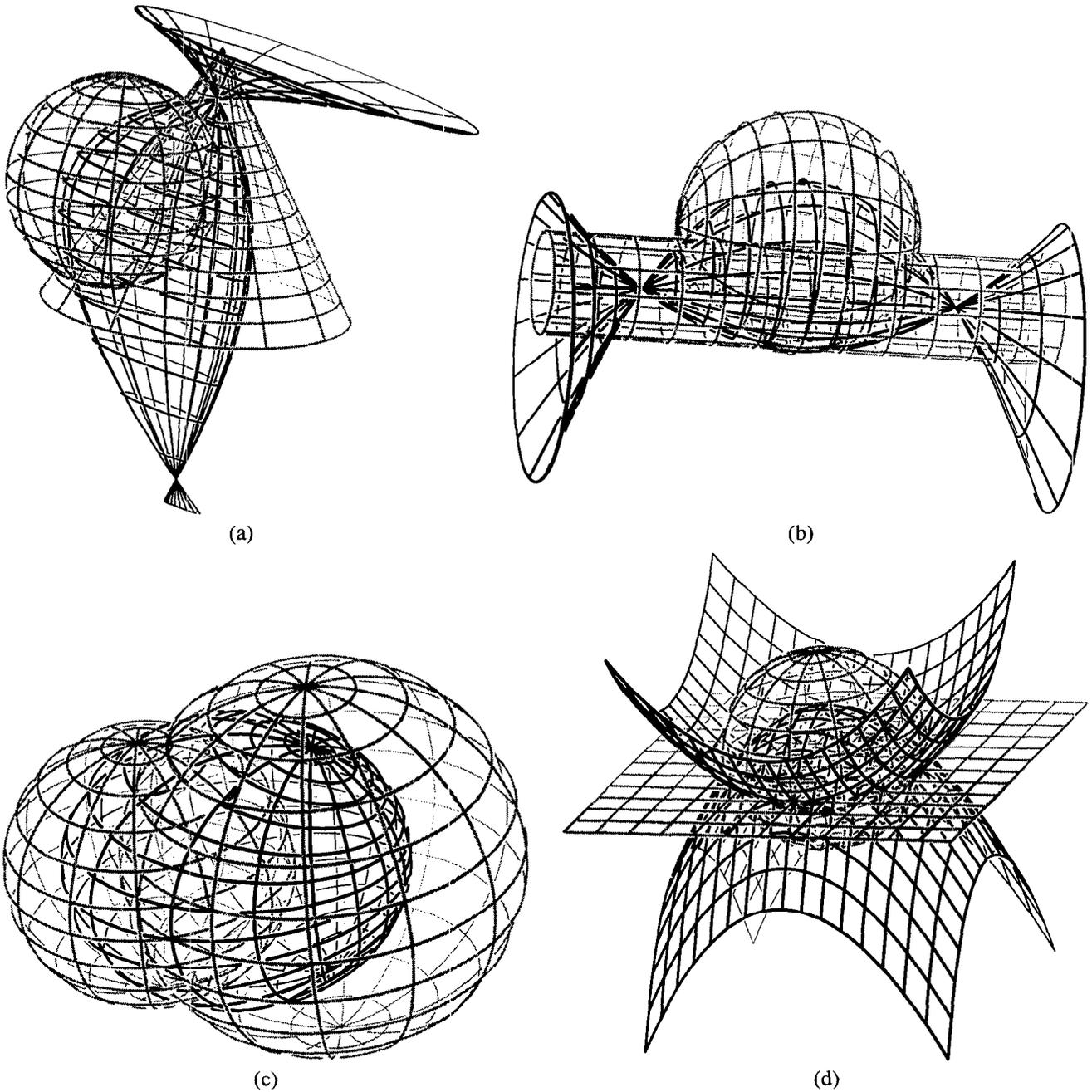


Figure 4: (a) shows the rational bisector (in black) of a cone and a sphere, (b) shows the rational bisector of a cylinder and a sphere, (c) shows the rational bisector of two spheres, and finally, (d) shows the rational bisector of a plane and a sphere.

	Point	Line	Plane	Cylinder	Sphere	Cone	Torus
Point	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Line		Yes	Yes	Partial (1)	Yes	Partial (2)	Partial (3)
Plane			Yes	Yes (4)	Yes (5)	Yes (6)	Partial (7)
Cylinder				Partial (8)	Yes (9)	Partial (10)	Partial (11)
Sphere					Yes (12)	Yes (13)	Yes (14)
Cone						Partial (15)	Partial (16)
Torus							Partial (17)

Table 1: A summary of special cases where the bisectors of CSG primitives can be represented as rational surfaces.

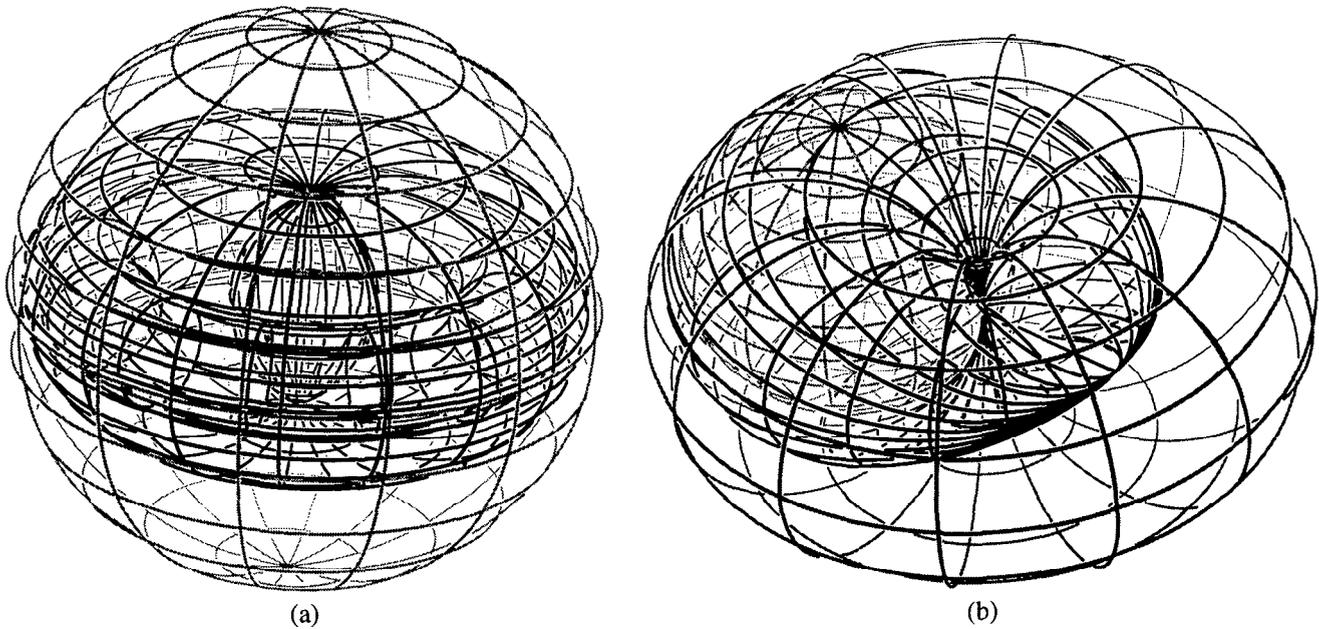


Figure 5: Two examples of rational bisector (in black) of a sphere (in gray) and a torus (in dark gray).

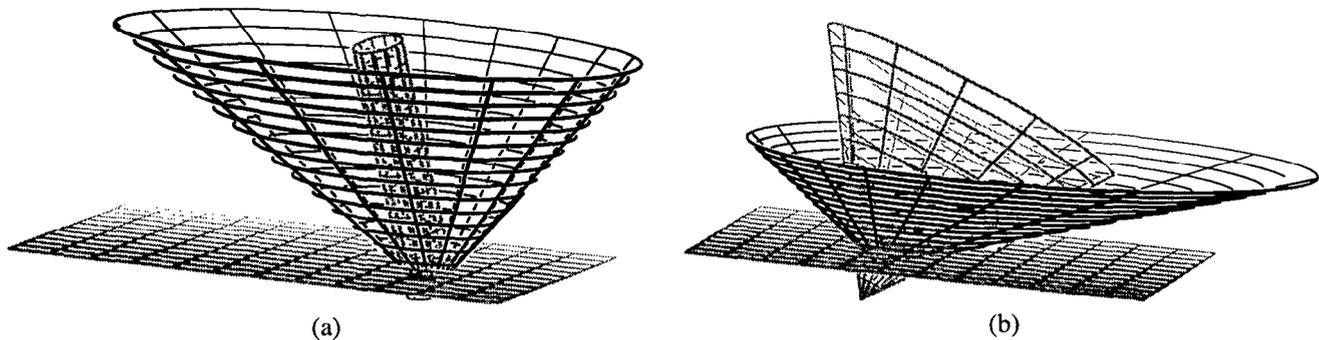


Figure 6: (a) shows a rational bisector of a cylinder and a plane, while in (b) a rational bisector of a cone and a plane is shown.

- (8) It is reducible to a cylinder-line bisector (see the case (1)). If the two cylinders are of equal radius, the problem is reducible to the line-line bisector. In this case, the bisector is a hyperbolic paraboloid if the axes are skew; a plane if the axes are parallel; and two planes if the axes intersect [1, 2].
- (9) A rational surface, via a reduction to a cylinder-point bisector [3].
- (10) If the axes are the same, the bisector consists of two cones [1].
- (11) If the axes are the same, the rational bisector is a surface of revolution with a parabolic cross section [1]. Alternatively, when the radius of the cylinder is the same as the minor radius of the torus, it is reducible to a circle-line bisector, which has a rational bisector [2].
- (12) A rational surface, via a reduction to a sphere-point bisector [3].
- (13) A rational surface, via a reduction to a cone-point bisector [3].
- (14) A rational surface, via a reduction to a torus-point bisector [3].
- (15) If two circular cones share the same apex, the problem can be reduced to a point-circle bisector on the sphere; thus the bisector is a rational conical surface with the same apex.
- (16) If the axes are the same, the bisector consists of two surfaces of revolution, each obtained by rotating a parabola about the common axis [1]. Note that the axis of rotation is neither identical nor perpendicular to the axis of symmetry of the parabola.
- (17) When the two tori have the same minor radius, the problem can be reduced to a circle-circle bisector. If the two circles are coplanar, the rational bisector is an elliptic or a parabolic cylinder. If the two circles are not coplanar, the bisector is also

a rational surface [2]. Alternatively, when the two tori have the same axis, the rational bisector is a surface of revolution with either an elliptic or a hyperbolic cross section [1].

## 5 Conclusion

In this paper, we have classified some special cases where the bisector surface of two CSG primitives can be represented as a rational surface in  $R^3$ . Dutta and Hoffmann [1] have already shown that many of these special bisectors are quadrics; thus, they can be represented as rational surfaces.

Herein, we have shown that some other cases are also rational, while not necessarily being quadrics. Our approach transforms each bisector problem to a simpler problem, i.e., either to a problem on the sphere or to that for simpler geometric primitives. As a result, we could add a few more special cases (of rational bisectors) to the result of Dutta and Hoffmann [1]. We also presented simple methods for constructing rational parameterization of several special bisector surfaces of CSG primitives.

Finally, it should be noted that the reductions to simpler configurations might eliminate some potential bisectors since the reductions based on offsets are dependent on surface orientation. Given two varieties, the reductions to simpler configurations might detect only subsets of the bisectors. Thus both inward and outward offsets must be considered for constructing complete bisectors. For example, two spheres with the same radius have a planar bisector between them. While this plane is typically the desired bisector, it is not the only solution. Figure 7 shows all potential bisectors of two spheres of the same radius. The inward reduction of these two spheres to two points would obviously yield the planar bisector. The reduction of one sphere to a point, while the other sphere is offset outward to a larger sphere, yields an ellipsoid.

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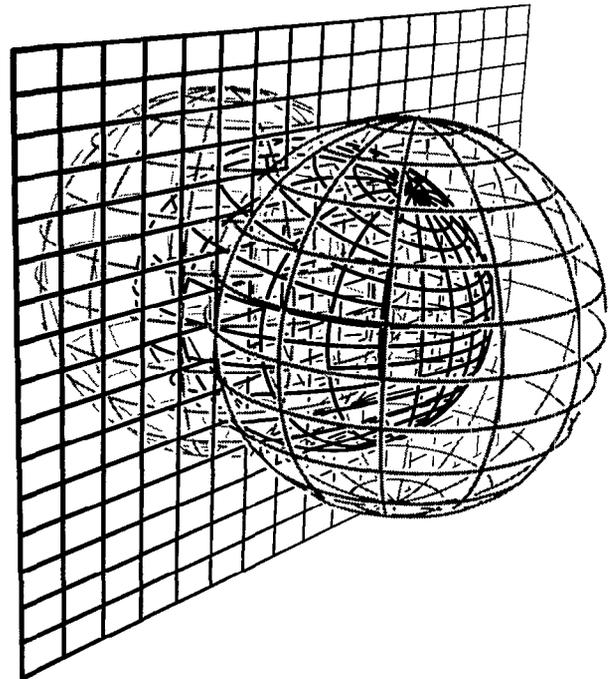


Figure 7: The bisectors (in black) between two spheres (in dark gray). The bisectors include a planar surface between the two spheres as well as an ellipsoid between the two.

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