6th Annual Video Review of Computational Geometry

The following notes give brief descriptions of the video segments that make up the accompanying video proceedings. As in previous years, the videos use images to enhance our understanding of geometric ideas. In some cases this involves the illustration of algorithms in action. In others, we see applications of geometric algorithms in other disciplines. As the field matures, we see the videos serving as valuable adjuncts in each of these functions.

The seven video segments were selected by a video program committee meeting in Vancouver on February 28th, 1997. The segments were judged based on their clarity, quality, and contribution to the field. Some accepted videos have been revised to reflect the comments of the committee.

We thank the members of the Video Program Committee for their help in evaluating the entries. The members of the committee were Alain Fournier (University of British Columbia), Mark Keil (University of Saskatchewan), David Salesin (University of Washington), Thomas C. Shermer (Simon Fraser University), and Jack Snoeyink (University of British Columbia). We thank the Computer Science Department at the University of British Columbia for providing support in the creation of the final video.

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The Bisector Surface of Freeform Rational Space Curves *

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Abstract

Given a point and a rational curve in the plane, their bisector curve is rational [3]. However, the bisector of two rational curves in the plane is not rational, in general [4]. Given a point and a rational *space* curve, the *bisector surface* is a rational ruled surface. Moreover, given two rational space curves, the bisector surface is rational (except for the degenerate case in which the two curves are coplanar). The last two cases are demonstrated in this video.
Key Words: Bisector, Voronoi surface, Medial surface, skeleton.

Given two objects in the plane or space, their bisector curve or surface is defined as the set of points which are equidistant from the two objects. While the bisector curve or surface can never self intersect, in this work, we consider rational representations of curves and surfaces that *contain* the exact bisector curve and/or surface. There are some redundant surface regions in the rational form that is presented that self intersect and hence must be eliminated. For brevity reasons, we denote in this work, those rational forms as the bisector curves and/or surfaces. The medial axis and medial surface are also closely related to the bisector curve and surface; that is, given an object in the plane or space, the medial axis or surface is defined as the set of interior points of the object which have minimum distance from at least two different points on the boundary of the object.

Bisector surfaces and medial surfaces have many important applications in engineering [1, 5, 6, 8]. (See Sherbrooke et al. [8] for a detailed survey.) However, their construction is non-trivial except for some special cases. Dutta and Hoffmann [1] considered the bisectors for simple surfaces such as natural quadrics and toroidal surfaces. For these special types of surfaces, the bisector surfaces have simple closed-form representations. For general algebraic surfaces, Hoffmann et al. [5, 6] formulated the bisector surface (called the Voronoi surface) using a simultaneous system of non-linear polynomial equations. The solution scheme is based on the *dimensionality paradigm* [5], which requires a preprocessing step that determines the global topological structure of the solution space.

Approximation of bisectors is computationally quite expensive. Therefore, it is desirable to classify special cases in which the bisectors have exact representations or simple approximations. Farouki and Johnstone [3, 4] investigated the bisector problem for planar rational curves. Given a point and a rational curve in the plane, the bisector curve is rational [3]. However, the bisector of two polynomial or rational curves in the plane is algebraic, but not rational, in general [4]. In practice, numerical tracing techniques are required to approximate the bisector curve. This video considers the bisector problem for space points and curves and shows that the added extra dimension of the 3D space actually alleviates the computational difficulty! That is, given a point and a space curve, their bisector is a rational ruled surface. Moreover, given two space curves, their bisector surface is rational (except for the two dimensional degenerate case in which the two curves are coplanar).

Given two C^1-continuous space curves, $C_1(s)$ and $C_2(t)$, consider the necessary conditions for the two points $C_1(s_0)$ and $C_2(t_0)$ to generate a bisector point P:

1. Point P must be located in the normal plane $L_1(s_0)$ of $C_1(s)$ at $C_1(s_0)$.
2. Similarly, point P must also be located in the normal plane $L_2(t_0)$ of $C_2(t)$ at $C_2(t_0)$.
3. Moreover, point P is also located in the bisector plane $L_{12}(s_0, t_0)$ which is the set of equidistant points from $C_1(s_0)$ and $C_2(t_0)$.

When the three planes: $L_1(s_0)$, $L_2(t_0)$, and $L_{12}(s_0, t_0)$, are in general position (i.e., no two of them are parallel to each other), there exists a unique intersection point P of the three planes.

When the curves $C_1(s)$ and $C_2(t)$ are rational 3-space curves, the planes $L_1(s)$, $L_2(t)$, and $L_{12}(s, t)$ can be represented as implicit equations (of x, y, z) with rational coefficients in s and t:

$$\begin{aligned}
L_1(s): &\quad a_1(s)x + b_1(s)y + c_1(s)z = d_1(s), \\
L_2(t): &\quad a_2(t)x + b_2(t)y + c_2(t)z = d_2(t), \\
L_{12}(s,t): &\quad a_{12}(s,t)x + b_{12}(s,t)y + c_{12}(s,t)z = d_{12}(s,t),
\end{aligned}$$

where all the coefficients are rational functions of s and t. Based on Cramer's rule, it is quite straightforward to show that x, y, z are rational functions of these coefficients; therefore, they are rational functions of s and t. As a result, we can represent the bisector surface $P(s, t)$ as a rational surface of s and t.

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When one of the two curves (say, $C_1(s)$) degenerates to a single point Q, the orthogonal plane $L_1(s)$ is not defined. Let $L_{12}(t_0)$ denote the bisector plane between points Q and $C_2(t_0)$. Then, we have the following plane equations of $L_2(t)$ and $L_{12}(t)$:

$$\begin{aligned} L_2(t): &\qquad a_2(t)x + b_2(t)y + c_2(t)z = d_2(t) \\ L_{12}(t): &\qquad a_{12}(t)x + b_{12}(t)y + c_{12}(t)z = d_{12}(t), \end{aligned}$$

where all the coefficients are rational functions of t. The common solution of the above two equations is the intersection line of two planes: $L_2(t)$ and $L_{12}(t)$. That is, each parameter t contributes a line to the bisector surface; thus the bisector surface becomes a ruled surface. The ruling direction $N(t)$ is given by the cross product of the normals of $L_2(t)$ and $L_{12}(t)$:

$$N(t) = (a_2(t), b_2(t), c_2(t)) \times (a_{12}(t), b_{12}(t), c_{12}(t)),$$

which is rational. To represent the ruled bisector surface as a rational surface, we need to construct a rational directrix curve on the surface. Let $L_1(t)$ be the plane which passes through the given point Q and is orthogonal to the ruled direction $N(t)$ at t:

$$L_1(t): \qquad a_1(t)x + b_1(t)y + c_1(t)z = d_1(t),$$

where $(a_1(t), b_1(t), c_1(t)) = N(t)$ and $d_1(t) = \langle N(t), Q\rangle$. All the coefficients of $L_1(t)$, $L_2(t)$, and $L_{12}(t)$ are rational; therefore, their common intersection point is also rational in t. The trace of these intersection points generates a rational directrix curve on the ruled bisector surface. Since the indicatrix curve $N(t)$ is also rational, the bisector surface is a rational ruled surface.

The computation of the bisector surface is efficient as it requires the symbolic solution of a small (3 by 3) linear system. The symbolic manipulation involves the summation and product of different piecewise polynomial and rational functions. Overall, and for all the bisector surfaces in this video, the computation of a single bisector surface on a high-end workstation takes a fraction of a second only. A wire-frame animation of the same video can, in fact, be computed and animated in real-time on a high-end workstation, at the rate of several frames per second.

Given two rational space curves, the existence of a rational bisector surface has great potential in surface design as well as in conventional engineering applications of medial surfaces. It is easy and quite intuitive to control the geometric shape of a (bisector) surface by changing the shape and orientation of the two base curves or to control the shape of a ruled (bisector) surface with one base curve and a base point. Motivated by the efficiency of the bisector surface computation, we propose the possibility of using the bisector surface construction as another freeform modeling construction scheme in traditional modeling environments, in a similar way to the sweep surface or surface of revolution construction schemes.

Clearly not all bisectors have rational representations. Questioning the varieties that can benefit from the presented symbolic approach is natural. Moreover, the extension of the proposed scheme to higher dimensions is feasible. In $\mathbb{R}^3$, the bisectors of the curve–point, curve–curve, and surface–point cases all have bivariate rational forms.

This video consists of a sequence of nearly 4000 frames, with each frame containing a bisector surface computed from different and continuously changing curves and points. The bisectors were computed using the symbolic tools of the IRIT [7] solid modeling system, developed at the Technion, Israel. The resulting bisector surfaces were then rendered transparently using a rendering tool of the same modeling system. Figure shows one example of a bisector surface computed to two quadratic Bspline space curves.

This text is an abstracted version of Reference [2].

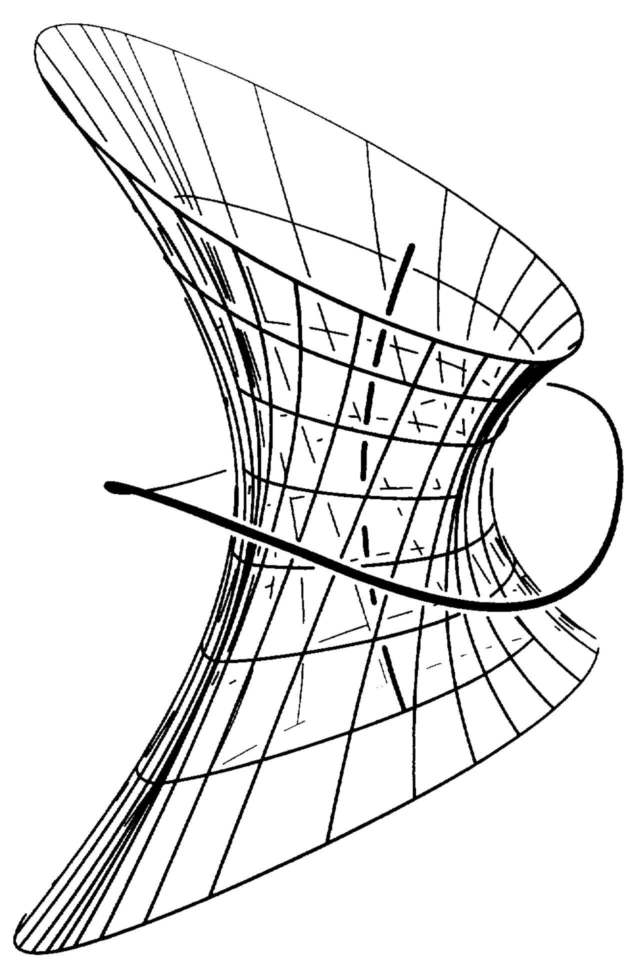

Figure 1: A bisector surface of two quadratic Bspline space curves. The bisector surface is order seven by seven.

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