Intersecting a Freeform Surface with a Ruled or a Ringed Surface

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Abstract

We present efficient and robust algorithms for intersecting a freeform surface with a ringed surface or a ruled surface. A ringed surface is given as a one-parameter family of circles. By computing the intersection between a freeform surface and each circle in the family, we can solve the intersection problem. We propose two approaches which are closely related to each other. The first approach detects certain critical points; and the intersection curve is constructed by connecting them in a correct topology. The second approach converts the intersection problem to that of finding the zero-set of two polynomial equations in the parameter space. The intersection between a freeform surface and a ruled surface can be computed in a similar way.

1. Introduction

Surface-surface intersection (SSI) is an important problem in geometric modeling and processing, particularly, for applications in CAD/CAM and solid modeling. Many different approaches have been proposed. However, it is still a difficult problem to solve with sufficient accuracy, efficiency and robustness. The main difficulty lies in analyzing the topological structure of the intersection curve; in particular, it is not easy to determine the exact number of connected components and the correct topological arrangement of these components.

Because of the difficulty in dealing with general freeform surfaces, considerable amount of research has been devoted to intersecting surfaces of special types [9]. Martin et al. [8] and de Pont [1] considered the intersection of a cyclide with a quadric or another cyclide. Johnstone [7] proposed an algorithm to intersect a cyclide with a ringed surface. (The term '*ringed* surface' was coined by Johnstone [7].) Heo et al. [5] have presented an algorithm for intersecting two ruled surfaces. They reduced the intersection problem to a search for the zero-set of the function: $f(u, v) = 0$, where $f(u, v)$ is a bivariate polynomial of relatively low degree. This approach was later applied to the case of intersecting two ringed surfaces [4]. In the present paper, we extend this result to the intersection of a freeform surface with a ringed surface or a ruled surface.

We present two methods for intersecting a freeform surface with a ringed surface (or a ruled surface). The first approach detects certain critical points on the intersection curve and connect them in a correct topology. The entire

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intersection curve is constructed using this procedure. The second approach transforms the intersection problem into the simpler problem of solving a system of two polynomial equations in three variables. (See Elber and Kim [2] or Patrikalakis and Maekawa [10] for more details of how to solve a system of m polynomial equations in n variables.)

The rest of this paper is organized as follows. In Section 2, we discuss how to construct the correct topology of an intersection curve based on detecting certain critical points on the curve. In Section 3, we give a simple technique for reducing the intersection problem to that of computing the simultaneous zero-set of two polynomial equations in three variables. Section 4 contains a discussion of the relationship between the two approaches. Finally, in Section 5, we conclude this paper.

2. Topology Construction

We will first present an algorithm that detects critical points on an intersection curve and constructs the correct topological structure of the curve. The basic idea of this approach is first explained for a simple case where a freeform surface is intersected with a cylindrical surface. Then we can proceed to more general cases where a freeform surface is intersected with a ruled surface or a ringed surface.

2.1. Intersection with a Cylindrical Surface

For the sake of simplicity, we first consider the case of intersecting a freeform surface and a cylindrical surface. We may assume that the cylindrical surface is generated by extruding a plane curve $C(t) = (a(t), b(t), 0)$ along the z-direction. Let $L(\mathbf{p})$ denote a line passing through a point $\mathbf{p}$ in the xy-plane and parallel to the z-axis. The line $L(C(t))$ intersects a freeform surface $S(u, v) = (x(u, v), y(u, v), z(u, v))$ tangentially if and only if the point $C(t) = (a(t), b(t), 0)$ is located on the silhouette of $S(u, v)$ when viewed along the z-direction. The silhouette curve and the boundary curve of $S(u, v)$ subdivide the xy-plane into several regions. Figure 1(a) shows the surface $S(u, v)$; and Figure 1(b) shows its silhouette and boundary curves projected on to the xy-plane. Also shown are the corresponding six regions $A_0, \cdots, A_5$ on the plane.

The location of a point $\mathbf{p}$ (in the xy-plane) leads to the following relation between a line $L(\mathbf{p})$ and the surface $S(u, v)$:

1. If $\mathbf{p}$ is on the silhouette curve of $S(u, v)$, the line $L(\mathbf{p})$ intersects $S(u, v)$ tangentially.
2. If $\mathbf{p}$ is inside a region A_i, then $L(\mathbf{p})$ intersects $S(u, v)$ transversally.

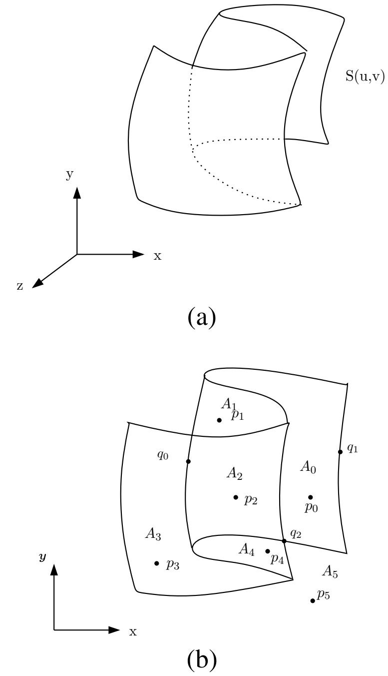


Figure 1. Regions on the xy-plane delimited by the silhouette curve and the boundary curve of $S(u, v)$.

3. If $\mathbf{p}$ and $\mathbf{q}$ are in the same region A_i, then $L(\mathbf{p})$ and $L(\mathbf{q})$ have the same number of intersections with $S(u, v)$.

For points $\mathbf{p}_i$, $0 \leq i \leq 5$, located in the region A_i (Figure 1(b)), the intersection between $L(\mathbf{p}_i)$ and $S(u, v)$ is classified as follows:

1. $L(\mathbf{p}_0) \cap S(u, v)$ consists of one regular point.
2. $L(\mathbf{p}_1) \cap S(u, v)$ consists of two regular points.
3. $L(\mathbf{p}_2) \cap S(u, v)$ consists of three regular points.
4. $L(\mathbf{p}_3) \cap S(u, v)$ consists of one regular point.
5. $L(\mathbf{p}_4) \cap S(u, v)$ consists of two regular points.
6. $L(\mathbf{p}_5) \cap S(u, v)$ has no intersection point.

For points $\mathbf{q}_j$, $0 \leq j \leq 2$, on the silhouette curve, the intersection between $L(q_j)$ and $S(u, v)$ is classified as follows:

1. $L(\mathbf{q}_0) \cap S(u, v)$ consists of one tangential intersection point and one regular intersection point.
2. $L(\mathbf{q}_1) \cap S(u, v)$ consists of one boundary point of $S(u, v)$.
3. $L(\mathbf{q}_2) \cap S(u, v)$ consists of one tangential intersection point and one boundary point of $S(u, v)$.

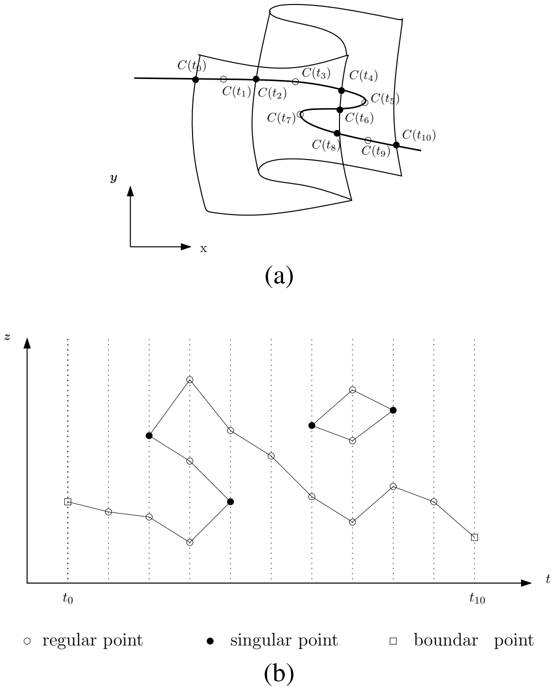


Figure 2. Topological structure of the intersection curve between $R(s,t)$ and $S(u,v)$.

Let $R(s,t)$ denote the cylindrical surface generated by extrusion of $C(t) = (x(t), y(t), 0)$ along the z-direction. The topological structure of the intersection curve between $S(u,v)$ and $R(s,t)$ is completely determined by the intersection between $C(t)$ and the silhouette and boundary curves of $S(u,v)$. By subdividing $S(u,v)$ and $C(t)$ if necessary, we may assume that the curve $C(t)$ does not pass through any of the self-intersection points of the silhouette curve of $S(u,v)$.

Figure 2 shows a sequence of points $C(t_i)$, for $i = 0, \cdots, 2m$, where $C(t_{2j})$ are the curve points on the silhouette or boundary of $S(u,v)$, and $t_{2j+1} = \frac{t_{2j}+t_{2j+2}}{2}$. Figure 2(b) shows the correct topology of the intersection curves which is completely determined by discrete intersection points $L(C(t_i)) \cap S(u,v)$, for $i = 0, \cdots, 2m$. The intersection points on each line $L(C(t_i))$ are sorted along the z-direction. We connect discrete points on two adjacent vertical lines according to their z-order. Each singular point is connected to two regular points on an adjacent vertical line; or it is an isolated point.

Three cases where the number of points on $L(C(t_{i+1}))$ is larger than that of $L(C(t_i))$ are shown in Figure 3. In Figure 3(a) there is a singular point (marked in bold type) on $L(t_{i+1})$, whereas in Figure 3(b) there is a boundary point (marked by a square) on $L(t_{i+1})$. In these cases, a new component starts at the point, and the other remaining regular points are connected as usual. Figure 3(c) shows the case where $L(t_i)$ contains a singular point, which may be considered as a double point and thus be connected to two regular points. In the alternative case, where the number of points on $L(t_i)$ is larger than the number on $L(t_{i+1})$, we apply a similar rule. In this case, two branches may meet at a singular point, where the corresponding loop is closed.

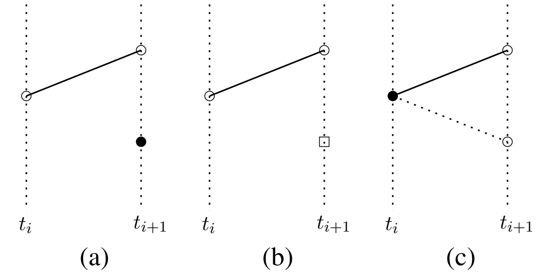


Figure 3. Three cases where there are more points on $L(t_{i+1})$ than on $L(t_i)$.

In the above discussion, we considered the case where the curve $C(t)$ intersects transversally the silhouette or boundary curve of $S(u,v)$. The case of tangential intersection is more involved. When the curve $C(t)$ intersects the silhouette curve of $S(u,v)$ tangentially at $\mathbf{p}$, we may consider it as double intersections. The tangential intersection point $L(\mathbf{p}) \cap S(u,v)$ becomes a singular point where two different loops meet and the intersection curve self-intersects. In case the curve $C(t)$ touches the silhouette curve of $S(u,v)$ externally, the singular point may be an isolated discrete point.

2.2. Intersection with a Ruled Surface

Now we consider the more general problem of intersecting a freeform surface $S(u,v)$ with a ruled surface $R(s,t)$. A ruled surface is defined by connecting two space curves $C_1(t)$ and $C_2(t)$ by a line:

$$R(s,t) = C_1(t) + s(C_2(t) - C_1(t)).$$

The ruling direction is no longer fixed, but is a function of t, namely $C_2(t) - C_1(t)$. Nevertheless, we can apply a similar argument to the one used in the previous section to classify the critical points on an intersection curve. So, we need to detect the values of t that correspond to the ruling lines (in $R(s,t)$) that intersect $S(u,v)$ tangentially or at its boundary curves $S(u_0, v)$ or $S(u, v_0)$. Figure 4 shows a loop in the intersection curve. The loop is delimited by two tangential intersections of the moving line with the surface $S(u,v)$.

When a ruling line is tangent to the surface $S(u,v)$, it is contained in the tangent plane of $S(u,v)$ and thus it is or-

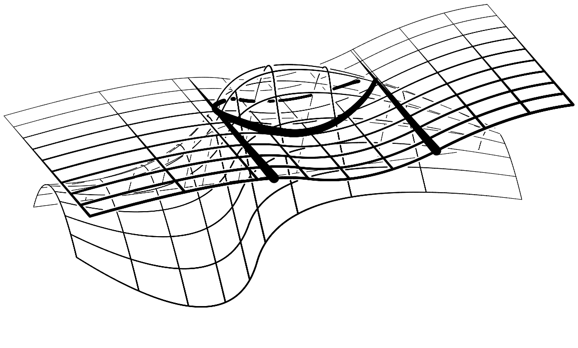


Figure 4. An intersection curve. Two ruling lines intersect tangentially with a freeform surface and delimit a loop on the intersection curve.

thogonal to the normal vector $N(u,v)$. Also, of course, the line and the surface have at least one point in common. (See Figure 5 for a configuration where a ruling line touches the surface $S(u,v)$.) From these conditions, we obtain the following system of three constraint equations:

$$\begin{aligned} f_1(u,v,t) &= \langle C_2(t) - C_1(t), N(u,v)\rangle = 0, && (1)\\ g_1(u,v,t) &= \langle S(u,v) - C_1(t), N(u,v)\rangle = 0, && (2)\\ h_1(u,v,t) &= \langle S(u,v) - C_1(t), N_1(t)\rangle = 0, && (3) \end{aligned}$$

where $N_1(t)$ is a vector which is orthogonal to the ruling direction $C_2(t) - C_1(t)$. In Section 3, we will explain a simple technique (originally proposed by Hughes and Möller [6]) that constructs two vectors $N_1(t)$ and $N_2(t)$ so that:

- They are orthogonal to the direction vector $C_2(t) - C_1(t)$.
- They are not parallel each other.
- Their degree is no higher than that of $C_2(t) - C_1(t)$.

Now the intersection between a boundary curve $S(u, v_0)$ and a ruling line can be computed by solving the following system of two equations:

$$\begin{aligned} f_3(u,t) &= \langle S(u,v_0) - C_1(t), N_1(t)\rangle = 0, && (4)\\ g_3(u,t) &= \langle S(u,v_0) - C_1(t), N_2(t)\rangle = 0. && (5) \end{aligned}$$

2.3. Intersection with a Ringed Surface

A ringed surface is a one-parameter family of circles, where the generator circle moves in space while continuously changing its position, size and orientation. Given a freeform surface $S(u,v)$ and a ringed surface $R(s,t) = \cup O_t$, their intersection curve can be computed by characterizing the intersection points between each circle O_t and the surface $S(u,v)$.

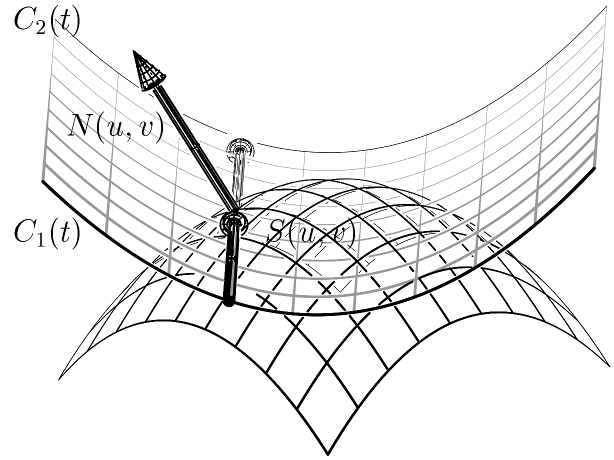


Figure 5. A line touches the surface $S(u,v)$.

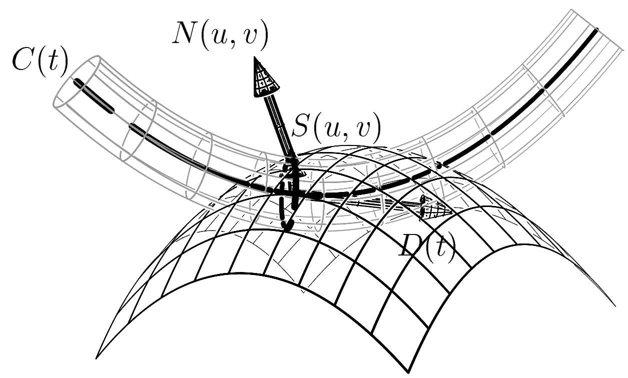


Figure 6. A circle touches the surface $S(u,v)$.

Let $C(t)$ denote the center of the circle O_t; and assume that the circle has radius $r(t)$ and is contained in a plane with normal $D(t)$. When the circle O_t intersects the surface $S(u,v)$ tangentially, the three vectors $S(u,v) - C(t)$, $D(t)$ and $N(u,v)$ become coplanar. (See Figure 6 for a configuration where a circle touches the surface $S(u,v)$.) Consequently, we have

$$f_2(u,v,t) = \langle (S(u,v) - C(t)) \times N(u,v), D(t)\rangle = 0. \quad (6)$$

Moreover, since the surface point $S(u,v)$ is contained in the circle O_t, we have

$$\begin{aligned} g_2(u,v,t) &= \langle S(u,v) - C(t), D(t)\rangle = 0, && (7)\\ h_2(u,v,t) &= \| S(u,v) - C(t) \|^2 - r^2(t) = 0. && (8) \end{aligned}$$

Equation (7) implies that $S(u,v)$ is contained in the plane of the circle O_t; and Equation (8) means that the circle O_t has radius $r(t)$.

Now the intersection between a boundary curve $S(u, v_0)$ and a circle O_t can be computed by solving the following system of two equations:

$$g_4(u,v,t) = \langle S(u,v_0) - C(t), D(t) \rangle = 0, \quad (9)$$
$$h_4(u,v,t) = \| S(u,v_0) - C(t) \|^2 - r^2(t) = 0. \quad (10)$$

2.4. Topology of the Intersection Curve

We first consider the intersection between a freeform surface and a ruled surface. After all critical points have been detected by solving the system of three polynomial equations in three variables, their t-values are sorted in ascending order with even indices: $t_0 < t_2 < \cdots < t_{2m}$. Now let $t_{2k+1} = \frac{t_{2k}+t_{2k+2}}{2}$. For each t_i, $(0 \le i \le 2m)$, we intersect the freeform surface $S(u,v)$ with a ruling line

$$R(s,t_i) : C_1(t_i) + s(C_2(t_i) - C_1(t_i)).$$

$R(s_j, t_i)$, $(1 \le j \le n_i)$, are the intersection points sorted along the ruling s-direction. We can now apply the topology construction scheme already shown in Figure 2. (But in this case the s-axis is used instead of the z-axis.) Once the topology is determined, each segment of the intersection curve is generated by numerically tracing along the intersection between two surfaces.

In the case of intersection with a ringed surface, the intersection points along the s-axis repeat themselves with a period of 2π. Thus the matching between two adjacent t_i-circles becomes a bit more complicated. However, a search coupled with numerical tracing will eventually find all the matching points.

3. Reduction to Parameter Space

We will now go on to present an alternative approach that reduces the intersection problem to that of computing the zero-set of two polynomial equations in three variables. The result is a 1-manifold in the uvt-parameter space. By projecting this 1-manifold on to the uv-parameter plane, the intersection curve can be constructed.

Figure 7 shows a ringed surface as a one-parameter family of circles, where each circle is defined as the intersection between a sphere and a plane. The intersection condition between a freeform surface $S(u,v)$ and a circle O_t is given as follows:

$$g_2(u,v,t) = \langle S(u,v) - C(t), D(t) \rangle = 0, \quad (11)$$
$$h_2(u,v,t) = \| S(u,v) - C(t) \|^2 - r(t)^2 = 0. \quad (12)$$

In the case of intersection with a ruled surface, we may reconstruct each ruling line

$$R(s,t) = C_1(t) + s(C_2(t) - C_1(t))$$

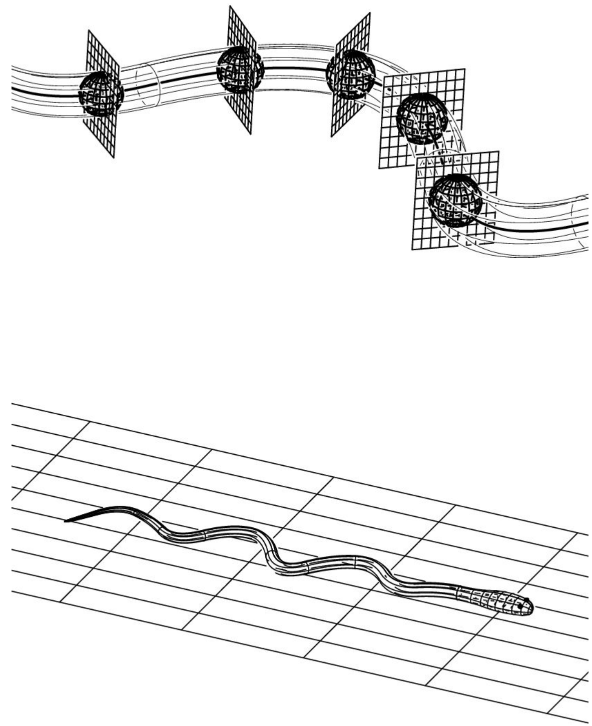


Figure 7. A ringed surface as a one-parameter family of circles, where each circle is defined as the intersection of a sphere and a plane.

as the intersection of two non-parallel planes with normals $N_1(t)$ and $N_2(t)$. (This technique was introduced by Huges and Möller [6].) By subdividing the ruled surface if necessary, we may assume that the ruling direction

$$C_2(t) - C_1(t) = (d_x(t), d_y(t), d_z(t))$$

satisfies the condition

$$|d_x(t)| \ge |d_y(t)|, |d_z(t)|.$$

The two normal vectors $N_1(t)$ and $N_2(t)$ can then be constructed as follows:

$$N_1(t) = (d_y(t), -d_x(t), 0),$$
$$N_2(t) = (d_z(t), 0, -d_x(t)).$$

Note that $N_1(t)$ and $N_2(t)$ are not parallel (see Figure 8).

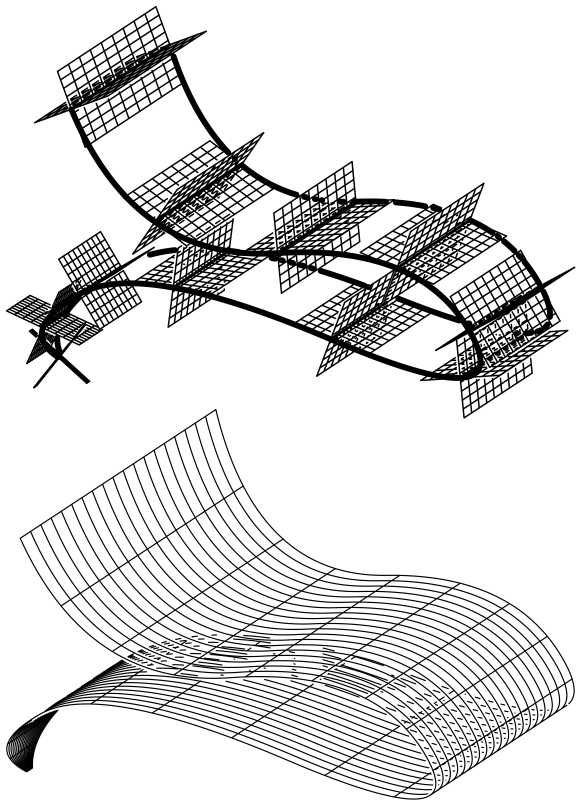


Figure 8. A ruled surface as a one-parameter family of lines, where each line is defined as the intersection of two non-parallel planes.

The intersection between a freeform surface $S(u,v)$ and the ruling line is characterized as follows:

$$f_3(u,v,t) = \langle S(u,v) - C_1(t), N_1(t) \rangle, \quad (13)$$
$$g_3(u,v,t) = \langle S(u,v) - C_1(t), N_2(t) \rangle. \quad (14)$$

Now we have reduced the intersection problem to that of computing the zero-set of two polynomial equations in uvt-parameter space. The result is a 1-manifold. By projecting this zero-set on to the uv-plane, the intersection is generated as a curve embedded on the freeform surface $S(u,v)$.

4. Comparison and Results

Assume that the freeform surface $S(u,v)$ is of degree (m,n) and the ruled or ringed surface is generated by space curves and radius functions of degree l. Table 1 shows the resulting degrees of the polynomial functions formulated in the previous sections.

The polynomial equations for topology construction have higher degrees than those for problem reduction.

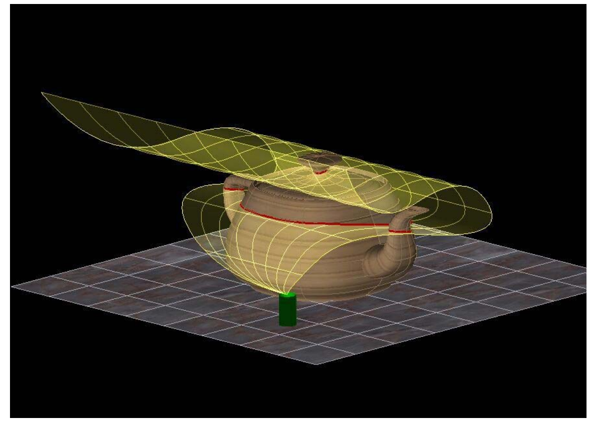


Figure 9. Intersection between the Utah teapot and a ruled surface.

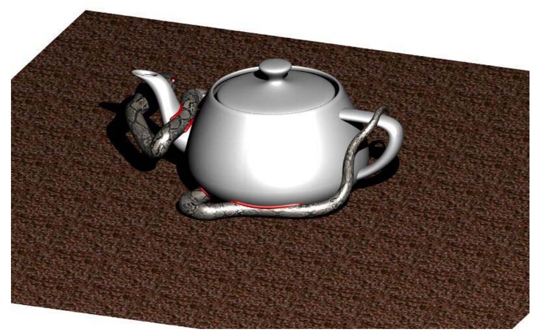


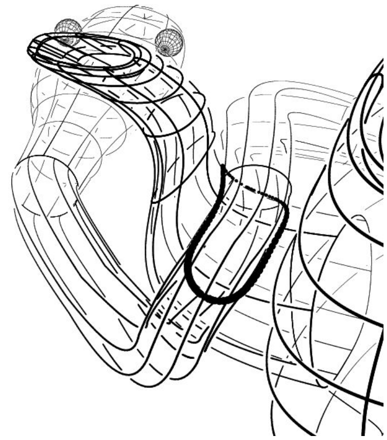
Figure 10. Intersection between the Utah teapot and a ringed surface.

Moreover, we have to solve three polynomial equations in topology construction, but just two in the reduction approach. However, the solutions of the three equations are discrete points, whereas the zero-sets of the two equations in the reduction approach are 1-manifolds. It is computationally more efficient to compute discrete solutions than to construct 1-manifolds. Thus it is worthwhile to formulate the relatively high-degree polynomial equations needed for topology construction.

Topology construction requires a second stage of numerical tracing that generates segments of the intersection curve. This can be achieved using conventional techniques for numerical tracing intersection curves. Alternatively, we can use the two polynomial equations in the uvt-space that have been formulated for the problem reduction approach.

	Topology Construction	Eqn. Number
Ruled Surface	$f_1 : 2m-1, 2n-1, l$	(1)
	$g_1 : 3m-1, 3n-1, l$	(2)
	$h_1 : m, n, 2l$	(3)
Ringed Surface	$f_2 : 3m-1, 3n-1, 2l$	(6)
	$g_2 : m, n, 2l$	(7)
	$h_2 : 2m, 2n, 2l$	(8)
	Problem Reduction	**Eqn. Number**
Ruled Surface	$f_3 : m, n, 2l$	(11)
	$g_3 : m, n, 2l$	(12)
Ringed Surface	$g_2 : m, n, 2l$	(9)
	$h_2 : m, n, 2l$	(10)

Table 1. Comparison between the degrees of the equations required in our two different approaches



These equations define two implicit surfaces in uvt-space and their normals can be represented using polynomials of lower degree than those required for the original surfaces. The two approaches proposed in this paper are thus closely related to each other.

Figure 9 shows the result of intersecting the Utah teapot with a ruled surface using the topology construction scheme. Figure 10 shows the same teapot intersected with a ringed surface. A result based on the simple reduction scheme is shown in Figure 11. The computation time for the results are similar. They are all within one or two seconds.

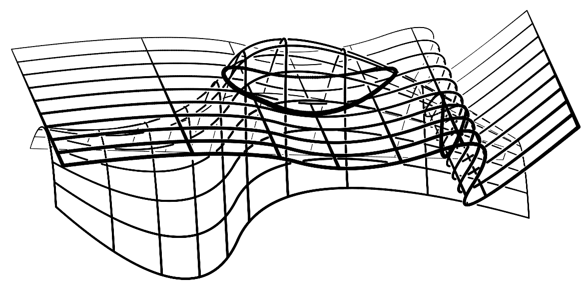


Figure 11. Intersection based on the problem reduction scheme.

5. Conclusion

We have introduced two approaches to the problem of intersecting a freeform surface with a ruled or a ringed surface. Our first scheme was based on detecting critical points and constructing the intersection curve with correct topology based on the critical points. Our second scheme reduces the SSI problem to a zero-set finding problem in uvt-space. These two schemes are closely related to each other. In future work, we plan to investigate more general classes of surfaces to which we can apply similar techniques.

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